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WORKSHOP ON KLEINIAN GROUPS AND RELATED TOPICS

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Departamento de Física y Matemáticas

Cuerpo Académico de Matemáticas Puras
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Presentación

El Cuerpo Académico de Matemáticas Puras y Aplicadas del Instituto de Ingeniería y Tecnología de la Universidad Autónoma de Ciudad Juárez (UACJ) fue creado en 2010 por la firme decisión institucional de consolidar el mejoramiento de la calidad de los académicos adscritos a la Licenciatura en Matemáticas del Departamento de Física y Matemáticas del Instituto de Ingeniería y Tecnología de la UACJ, mediante la incorporación de sus miembros en actividades de investigación científica colectiva y con el compromiso de transmitir esta generación de conocimiento a toda la comunidad universitaria.

Las Líneas de Generación y Aplicación del Conocimiento que cultiva el Cuerpo Académico de Matemáticas Puras y Aplicadas son: Álgebra, Ecuaciones diferenciales y Sistemas dinámicos, matemáticas aplicadas y lógica matemática y fundamentos. Sus miembros trabajan de manera colegiada para la creación de nuevos conocimientos en estas áreas y sus aplicaciones. Asimismo, se busca involucrar a estudiantes avanzados de la Licenciatura en Matemáticas en estas actividades con el objetivo de acercarlos a la investigación científica, contribuyendo con ello a una formación más sólida de los egresados de esta licenciatura.

El presente material es un registro de algunas de las actividades realizadas por el Cuerpo Académico de Matemáticas Puras y Aplicadas. Todos los trabajos incluidos, se han presentado en las actividades regulares del Cuerpo Académico o son producto de los esfuerzos colectivos de sus miembros y sus relaciones de colaboración con investigadores de otras instituciones.

Luis Loeza Chin
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Limit set of cyclic subgroups of $\mathrm{PSL}(3,\mathbb{C})$ *

Adriana González Urquiza †

Abstract

Let $\mathbb{P}^2_{\mathbb{C}}$ be the complex projective space. It is known that each biholomorphism $f:\mathbb{P}^2_{\mathbb{C}}\to\mathbb{P}^2_{\mathbb{C}}$ is either loxodromic, parabolic or elliptic [8]. Let $\mathrm{PSL}(3,\mathbb{C})$ be the group of all biholomorphisms of $\mathbb{P}^2_{\mathbb{C}}$. We use quasi-projective transformations to describe the limit set, as defined by Kulkarni, of the cyclic subgroups of $\mathrm{PSL}(3,\mathbb{C})$.

Let $\mathrm{PU}(2,1)$ be the group of all isometries of the two dimensional hyperbolic space $\mathbf{H}^2_{\mathbb{C}}$. We describe the limit set of subgroups of $\mathrm{PU}(2,1)$ acting on the complex projective space. In particular, we show that the Kulkarni limit set is the union of all tangent lines to the hyperbolic plane $\mathbf{H}^2_{\mathbb{C}}$, in points of the Chen-Greenberg limit set.

The results we present in this note, continues the study iniciated by Juan Pablo Navarrete in [7] and [8].

Keywords: Limit set, Kleinian groups, classification of isometries.

1 Introduction

The group $\mathrm{PSL}(2,\mathbb{C})$ is the group of isometries of the complex one-dimesional hyperbolic space $\mathbf{H}^1_{\mathbb{C}}$, using the model of the Poincaré disk. For the elements in $\mathrm{PSL}(2,\mathbb{C})$ there is a classification according to their dynamics. The classification is given algebraically by the trace of the lifting in $\mathrm{SL}(2,\mathbb{C})$ or by the fixed points that the transformation has. It is, therefore, natural to wonder about the classification of the isometries of complex hyperbolic spaces of higher dimensions.

In this way, William Goldman [4] made the classification of isometries of $\mathbf{H}^2_{\mathbb{C}}$, that is, the elements of the group $\mathrm{PU}(2,1)$. Juan Pablo Navarrete [8] extended the classification to the transformations of $\mathrm{PSL}(3,\mathbb{C})$, which is the group whose elements are all the biholomorphisms of the complex projective plane. Also studied in his paper, are the limit sets of the cyclic subgroups generated by one element in $\mathrm{PSL}(3,\mathbb{C})$. The author uses the Bergman metric in the space to prove that there are sequences that converge to the elements in the limit set.

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Navarrete in [7] establishes the relationship between the Kulkarni limit set and the Chen-Greenberg limit set of a subgroup of PU(2, 1).

When the complex Kleinian groups began to be studied, nothing was known about the limit sets, so Navarrete's discovery was enlightening, although the approach to finding the limit sets of groups becomes inefficient when calculating the limit set of subgroups of automorphisms of higher dimension spaces.

In this paper, we use *quasi-projective transformations* as another way of finding the limit set of subgroups of $\mathrm{PSL}(3, \mathbb{C})$; additionally, these transformations are useful for establishing a relationship between the Kulkarni limit set and the Chen-Greenberg limit set of a subgroup of $\mathrm{PU}(2, 1)$. This tool provides a simpler method for calculating the limit set of subgroups of automorphisms of higher dimension spaces.

2 Preliminaries

Recall that $\mathrm{SL}(3, \mathbb{C})$ is the group of matrices 3×3 with coefficients in $\mathbb{C}$ and determinant one. Due to the fact that for $A \in \mathrm{SL}(3, \mathbb{C})$, $\det(\lambda A) = \lambda^3 \det(A)$, we have three different matrices representing the same biholomorphism. Let $\{1, \omega, \omega^2\}$ be the cubic roots of unity, it is defined $\mathrm{PSL}(3, \mathbb{C})$ as $\mathrm{SL}(3, \mathbb{C})/\{1, \omega, \omega^2\}$, and the elements in $\mathrm{PSL}(3, \mathbb{C})$ are the biholomorphisms of the complex projective plane $\mathbb{P}^2_{\mathbb{C}}$.

2.1 Kulkarni's Limit Set

Recall that for $x \in X$, the orbit of x is $o(x, G) = \{y = g(x) | g \in G\}$, and for a compact subset $A \subset X$, the $G-$orbit of A is $\{g(A) | g \in G\}$.

Definition 2.1. Let X be a locally compact Hausdorff space with a countable base for its topology. Let G be a group acting on X and let $\Omega \subset X$ be a G-invariant subset. The action on Ω is properly discontinuous if for every pair of compact subsets C and D of Ω, the cardinality of the set $\{g \in G | g(C) \cap D \neq \varnothing\}$ is finite.

In 1978, Ravi Kulkarni [6] defined a limit set. Let G be a subgroup of $\mathrm{PSL}(3, \mathbb{C})$, the *limit set in the sense of Kulkarni* as the union of three subsets:

$$\Lambda(G) = L_0(G) \cup L_1(G) \cup L_2(G), \tag{1}$$

where each subset is given by:

$L_0(G)$ the closure of points in $\mathbb{P}^2_{\mathbb{C}}$ with infinite isotropy group,

$L_1(G)$ the closure of accumulation points of the orbits of points in $\mathbb{P}^2_{\mathbb{C}} \setminus L_0(G)$,

$L_2(G)$ the closure of acummulation points of $G-$orbits of compact subsets contained
in $\mathbb{P}^2_{\mathbb{C}} - (L_0(G) \cup L_1(G))$.

The complement of this union is the *discontinuity region*, and it is denoted by $\Omega(G)$. We work with this definition throughout this paper.

2.2 Quasi-Projective Transformations

In this work, the *quasi-projective transformations* will be very important to describe the limit set of subgroups of automorphisms of $\mathbb{P}^2_{\mathbb{C}}$. We introduce them.

Let $M : \mathbb{C}^3 \to \mathbb{C}^3$ be a nonzero linear transformation, and $\mathbb{C}^* = \mathbb{C} - \{0\}$. Observe that M is not necessarily invertible. Consider the kernel of the transformation $\ker(M) \subset \mathbb{C}^3$. We denote by $[\,] : \mathbb{C}^3 - \{0\} \to \mathbb{P}^2_{\mathbb{C}}$ the canonical projection. Let $[\ker(M)] \subset \mathbb{P}^2_{\mathbb{C}}$ be the projectivization of $\ker(M)$. Precisely, $[\ker(M)] = \ker(M)/\mathbb{C}^*$. M induces a well-defined transformation $\widetilde{M} : \mathbb{P}^2_{\mathbb{C}} - [\ker(M)] \to \mathbb{P}^2_{\mathbb{C}}$, given by

$$\widetilde{M}([v]) = [M(v)].$$

Indeed, it is well defined because $M(v) \neq 0$, and it is a projective transformation on its domain: for every $\lambda \in \mathbb{C}^*$, $\widetilde{M}[\lambda v] = [\lambda M(v)]$, coinciding with $[M(v)]$ in $\mathbb{P}^2_{\mathbb{C}}$.

We call $\widetilde{M}$ a *quasi-projective transformation*, and we denote the set of these transformations of $\mathbb{P}^2_{\mathbb{C}}$ as $QP(3,\mathbb{C})$; this space is the closure of $\mathrm{PSL}(3,\mathbb{C})$. It is known that the pointwise convergence is equivalent to the convergence as a space of transformations. The following proposition is shown in [2].

Proposition 2.2. *Let $(g_m)_{m\in\mathbb{N}} \subset PSL(3,\mathbb{C})$ be a sequence of distinct elements; then there exists a subsequence, still denoted $(g_m)_{m\in\mathbb{N}}$ and a transformation $g \in QP(3,\mathbb{C})$, such that $g_m \to g$ when $m \to \infty$ in compact subsets of $\mathbb{P}^2_{\mathbb{C}} - [\ker(g)]$.*

We will use this transformations after noting that there is at least a lifting for each transformation in $\mathrm{PSL}(3,\mathbb{C})$; we can, in fact, multiply g^n by a scalar $\alpha \in \mathbb{C}^*$, and, after projecting, we get the same transformation. If multiplied by an adequate scalar, the sequence $[\alpha_n g^n]$ converges to a quasi-projective transformation.

2.3 Quasi-minimality Lemma

This lemma is relevant when we calculate the limit set of cyclic groups. It is introduced in [8].

Lemma 2.3. *Let G be a subgroup of $PSL(3,\mathbb{C})$. If $C \subset \mathbb{P}^2_{\mathbb{C}}$ is a closed subset, such that, for every compact subset $K \subset \mathbb{P}^2_{\mathbb{C}} \setminus C$, the accumulation points of the family $\{g(K)\}_{g\in G}$ are contained in $L_0(G) \cup L_1(G)$, then $L_2(G) \subseteq C$.*

Proof. Suppose there is a point $X \in L_2 - C$. By the definition of L_2, X is in the closure of the accumulation points of the orbit of some compact subset in $\mathbb{P}^2_{\mathbb{C}} - L_0 \cup L_1$.

Let (k_m) be a sequence in $\mathbb{P}^2_{\mathbb{C}} - L_0 \cup L_1$, such that k_m converges to $k \in \mathbb{P}^2_{\mathbb{C}} - L_0 \cup L_1$. And suppose that there is a subsequence $(g_m) \subset G$, such that $g_m(x_m) \to X \in \mathbb{P}^2_{\mathbb{C}} - C$; as the hypotheses of the lemma are valid for compact sets in $\mathbb{P}^2_{\mathbb{C}} - C$. Then there is $N \in \mathbb{N}$, such that for every $m > N$, $g_m(k_m) \in \mathbb{P}^2_{\mathbb{C}} - C$. Therefore, $\{g_m(x_m)\} \cup \{X\}$ is a compact set in $\mathbb{P}^2_{\mathbb{C}} - C$.

According to the hypothesis, the accumulation points of compact subsets are contained in $L_0 \cup L_1$. If we consider $g_m^{-1}(g_m(k_m)) = k_m \longrightarrow k$, we attain a contradiction because we started the proof assuming that k is not an element of $L_0 \cup L_1$. □

2.4 Notation

Let e_1, e_2 and e_3 be the projectivization of the canonical basis of $\mathbb{C}^3$. We denote by $\mathrm{Diag}(a, b, c)$ the diagonal matrix, whose values in the diagonal are a, b and c.

Recall that the set of accumulation points of any set A is denoted A'.

3 Limit sets of cyclic subgroups

We proceed with the calculation of the limit sets of different transformations.

3.1 Loxodromic Transformations

We define the *loxodromic* elements in $\mathrm{PSL}(3,\mathbb{C})$ as those that have a lift in $SL(3,\mathbb{C})$, whose Jordan canonical form is one of the following matrices:

$$(1.a)\quad g = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix}, \quad (1.b)\ g = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_1 & 0 \\ 0 & 0 & \alpha_2 \end{bmatrix}, \quad (1.c)\ g = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{bmatrix},$$

where $|\alpha_i| < |\alpha_3|$, $i < 3$; where $|\alpha_1| \neq |\alpha_2|$; where $|\lambda| \neq 1$.

In order to describe the limit set of this type of transformation, we first find the points with infinite isotropy group. Let $X = [x : y : z]$ be any point in $\mathbb{P}^2_{\mathbb{C}}$; the points we are looking for are those which satisfy the next equation:

$$g^n(X) = [\alpha_1^n x : \alpha_2^n y : \alpha_3^n z] = [x : y : z]. \tag{2}$$

The solutions of this equation are e_1, e_2 and e_3. Thus, $L_0 = \{e_1, e_2, e_3\}$.

To find the set $L_1(G)$, observe that $\alpha_3^{-n} g^n$ converges to the matrix $Diag(0,0,1)$ as n goes to infinity; therefore, e_3 is an attracting point for $X \neq e_1, e_2, e_3$.

But if $|\alpha_1| < |\alpha_2|$, then $[g^{-n}] = [\alpha_1^n g^{-n}]$ converges to the transformation given by the matrix $Diag(1,0,0)$ as n goes to infinity; then e_1 is a repelling accumulation point.

We multiply the n-th power of the transformation by α_2^{-n}. Applying these iterates to the points in the line $\{z = 0\}$, they converge to $[e_2]$ as n goes to infinity. And if we apply the $n-$power of g^{-1} to the points in the line $\{[X = 0]\}$, the sequence $[g^{-n}] = [\alpha_2^n g^{-n}] \to e_2$. So, $L_1 = \{e_1, e_2, e_3\}$.

In order to know L_2, we have to prove two contentions. And using lemma 2.3, the proof would be complete:

Claim 3.1. $L_2 \subseteq \overleftrightarrow{[e_1 e_2]} \cup \overleftrightarrow{[e_2 e_3]}$

Proof. Take the union of these lines as the closed subset of lemma 2.3. We already know that all the points outside these lines converge to the future to $[e_3]$ and to the past to $[e_1]$, and both points are part of $L_0(g) \cup L_1(g)$. □

On the other hand, take a line ℓ, such that $e_1, e_2, e_3 \notin \ell$, and we can prove the next claim.

Claim 3.2. If the group $G = \langle g \rangle$ is generated by an element as in (1.a), then either the line $\overleftrightarrow{[e_2][e_3]}$ or $\overleftrightarrow{[e_1][e_2]}$ lies in the limit set $\Lambda(G)$.

Proof. If there exists a point $p \in \overleftrightarrow{[e_1][e_2]}$, such that $p \notin \Omega'$, then there is a neighborhood $B_r(p)$, such that $B_r(p) \notin \Omega'$. Take a point $q \in \overleftrightarrow{[e_2][e_3]} \setminus \{e_2, e_3\}$. And build the line $\overleftrightarrow{pq} = \ell$. Apply g^n to the line ℓ; the image of this line is again a line in $\mathbb{P}^2_{\mathbb{C}}$. The line ℓ and each of its images defines a point in $(\mathbb{P}^2_{\mathbb{C}})^*$, and so, we have a sequence of points in $(\mathbb{P}^2_{\mathbb{C}})^*$.

The line $\overleftrightarrow{pq} \in \mathbb{P}^2_{\mathbb{C}}$ defines a point $h_0 \in (\mathbb{P}^2_{\mathbb{C}})^*$. Moreover, $g^n(\overleftrightarrow{pq}) = \overleftrightarrow{g^n(p) g^n(q)}$. And as $g^n(p) \to e_2$ and $g^n(q) \to e_3$, then $\overleftrightarrow{g^n(\ell)} \to \overleftrightarrow{[e_2][e_3]}$.

Up to here, for all $y \in \overleftrightarrow{[e_2][e_3]}$ there is a sequence of points $(x_n) \subset \overleftrightarrow{pq}$, such that $g^n(x_n)$ converges to y. We claim that the sequence of points (x_n) converges to q. This happens because if it would converge to another point $x \neq q$, that point would satisfy: $x \notin \overleftrightarrow{[e_2][e_3]}$, due to the uniqueness of q in the intersection of this line with ℓ.

Then, the dynamic would be:

$$g^n(x_n) \to g^n(x) \to [e_3]. \tag{3}$$

Which is a contradiction, because that sequence converges to y.

The compact subset $(x_n) \cup \{x\}$ converges to $y \in \overleftrightarrow{[e_2][e_3]}$; therefore, the entire line $\overleftrightarrow{[e_2][e_3]}$ is in Ω'.

Through an analogous reasoning it is shown that each point in the line $\overleftrightarrow{[e_1][e_2]}$ is also an accumulation point and is in the limit set of the group $G = \langle g \rangle$. □

Finally, the result which is proven with the analysis above and the claims 3.1 and 3.2 is the following:

Proposition 3.3. *The Kulkarni limit set for $G = \langle g \rangle$ is $\Lambda(G) = \overleftrightarrow{[e_1][e_2]} \cup \overleftrightarrow{[e_2][e_3]}$.*

The transformations of type (1.a) are called *strongly loxodromic* whenever $|\alpha_1| < |\alpha_2|$. If this last condition doesn't hold, it is only called *loxodromic*. We can add the conditions: $|\alpha_1| = |\alpha_2| \neq |\alpha_3|$; so $\alpha_1/\alpha_2 = e^{2\pi i a}$; if $a \in \mathbb{Q}$, this map is called a *rational screw*, but if $a \in \mathbb{R} - \mathbb{Q}$, the transformation is an *irrational screw*.

We check the case $a \in \mathbb{Q}$. Again, the points e_1, e_2 and e_3 are fixed points; even more, the points on the line $\overleftrightarrow{e_1 e_2}$ have infinite isotropy group, due to the equality $[g] = [\alpha_2^{-1} g] = Diag(e^{2\pi i a}, 1, \alpha_2^{-1}\alpha_3)$ as a is rational, for infinitely many $n \in \mathbb{N}$, $e^{2\pi i n a} = 1$. Then $L_0 = \overleftrightarrow{e_1 e_2} \cup \{e_3\}$.

If $|\alpha_3| < |\alpha_2|$, we multiply $[\alpha_2^{-n} g^n]$, this transformation converges to $Diag(1, 1, 0)$ as n goes to infinity; therefore, the points in $\mathbb{P}^2_{\mathbb{C}}$ different from e_1, e_2, e_3 converge to the line $\overleftrightarrow{e_1 e_2}$. Calculating the limit as $n \to \infty$, $\alpha_3^n g^n(X)$ goes to $[e_3]$. And we have $L_1 = L_0$.

By lemma 2.3, $L_2(g) \subset \overleftrightarrow{e_1 e_2} \cup \overleftrightarrow{e_2 e_3}$. To prove the converse, we use a reasoning analogous to the one in claim 3.2.

When $a \in \mathbb{R} - \mathbb{Q}$, the only change is that $L_0 = \{e_1, e_2, e_3\}$.

Type (1.b)

A transformation as in $(1.b)$ is called *complex homothety*. The fixed points of this transformation are the ones in the line $\{z = 0\}$ and e_3; and they are also the only points with infinite isotropy group, so $L_0 = \overleftrightarrow{e_1 e_2} \cup [e_3]$. Without loss of generality, suppose $|\alpha_1| < |\alpha_2|$. The sequence $\alpha_2^{-n} g^n$ converges to the transformation $Diag(0, 0, 1)$ as n goes to infinity; therefore, $[e_3]$ is an attracting point. Also $\alpha_1^n g^{-n}$ converges to $Diag(1, 1, 0)$ if n goes to infinity; that's why we conclude that $\overleftrightarrow{e_1 e_2}$ is a line of repelling points.

If a compact subset is outside $\overleftrightarrow{e_1e_2}\cup\{e_3\}$, the images of this compact subset under the action of the group converge to $[e_3]$ to the future and to $\overleftrightarrow{e_1e_2}$ to the past. Therefore, $L_0 = L_1 = L_2 = \Lambda = \overleftrightarrow{e_1e_2}\cup\{e_3\}$.

Type (1.c)

For these transformations, the fixed points are $\{e_1, e_3\}$ and only they belong to the set L_0. Furthermore, e_3 is an accumulation point of orbits of points outside the line $\overleftrightarrow{e_1e_2}$; e_1 is an accumulation point of orbits of g^{-n} for points different from e_3. The orbit of the points of the line $\overleftrightarrow{e_1e_2}$, accumulate in e_1. Then $L_1 = \{e_1, e_3\}$. To calculate L_2, we make reference again to the claim 3.2; and it is formed by the lines $\overleftrightarrow{e_1e_2}\cup\overleftrightarrow{e_1e_3}$.

3.2 Parabolic Transformations

In this section, we make the analysis of the limit set of the cyclic group generated by a parabolic transformation.

If the element is *parabolic*, then it has a lifting, whose Jordan canonical form is given by one of the following matrices:

$$2.a)\quad g=\begin{bmatrix}1&1&0\\0&1&0\\0&0&1\end{bmatrix},\quad 2.b)\quad g=\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix},\quad 2.c)\quad g=\begin{bmatrix}1&1&0\\0&1&0\\0&0&\lambda\end{bmatrix},$$

where $\lambda = e^{2\pi i a} \neq 1$.

Type (2.a)

The points with infinite isotropy are the points in the line $\overleftrightarrow{[e_1e_3]}$, because the transformation on this line is the identity, obtaining L_0.

To find L_1, we look at the iterations of $(2.a)$ and get as limit transformation:

$$g^n=\begin{bmatrix}1&n&0\\0&1&0\\0&0&1\end{bmatrix}\longrightarrow h=6\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}. \tag{4}$$

That is, the sequence of transformations converges to the constant transformation $[e_1]$ if we iterate forward (and also if we iterate backward) for points outside $\overleftrightarrow{e_1e_3}$. Observe that in this particular case $\overleftrightarrow{e_1e_3}$ is precisely the subspace that must be ommited when calculating $L_1(G)$, since the points in the line are fixed. So $L_1(G) = \{e_1\}$.

The set L_2 is composed by the accumulation points of the compact sets in $\mathbb{P}^2_{\mathbb{C}} - \overleftrightarrow{[e_1e_3]}$. If U is an open set that contains e_1 and K is a compact subset outside $\overleftrightarrow{e_1e_3}$, by proposition 2.2 there exists $N \in \mathbb{N}$ such that for every $n > N$, then $g^n(K) \subset U$.

Then $L_2(G) = \{e_1\}$, and the Kulkarni limit set $\Lambda(G)$ of the cyclic group G is $\overleftrightarrow{[e_1e_3]}$.

Type (2.b)

Applying the parabolic transformation in $(2.b)$ to a point $[x : y : z]$ in $\mathbb{P}^2_{\mathbb{C}}$, we have that the only point with infinite isotropy group is e_1; it is, in fact, a fixed point. Moreover, $\overleftrightarrow{[e_1e_2]}$ is invariant under g.

In order to find the closure of the accumulation points of orbits of elements in $\mathbb{P}^2_{\mathbb{C}} - \{e_1\}$, the process is the same as in the previous example, that is, iterating the transformation and dividing by the entry of greater norm in g^n, and then as n goes to infinity, the limit transformation is:

$$h = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & h \end{bmatrix}.$$

If we apply h to a point $[x : y : z]$ outside $\overleftrightarrow{e_1e_2}$, its image is the point $[e_1]$, and follows that it is an attracting point. Doing the same to the inverse transformation, the image of the limit transformation is $[e_1]$; then it is a repelling point. However, in this case, $L_0(G)$ does not coincide with the kernel of h. Yet, we have to verify what happens to points in $\overleftrightarrow{e_1e_2}$.

This line is invariant under g and the transformation restricted to $\overleftrightarrow{e_1e_2}$ has a parabolic behavior, that is, any point on the line converges to the fixed point $\{e_1\}$. Therefore, $L_1(G) = \{e_1\}$.

To find the invariant lines is enough to compare the vectors of the form $(x + y, y + z, z)$ with $(\alpha x, \alpha y, \alpha z)$ for $\alpha \in \mathbb{C}^*$. There is an invariant line: $\overleftrightarrow{[e_1][e_2]}$.

Here, again, lemma 2.3 is used to show that $L_2(G) \subseteq \overleftrightarrow{[e_1][e_2]}$. And another lemma is required; this lemma is found in [8].

Lemma 3.4. *Let K be a compact subset of $\mathbb{P}^2_{\mathbb{C}} \setminus \overleftrightarrow{[e_1e_2]}$, and g as in $(2.b)$, then $[e_1]$ is the only accumulation point of the family $\{g^n(K)\}_{n\in\mathbb{Z}}$.*

Claim 3.5. The line $\overleftrightarrow{[e_1][e_2]}$ is part of the limit set of the group $G = \langle g \rangle$.

Proof. Let L be the line in $(\mathbb{P}^2_{\mathbb{C}})^*$, the dual space of the complex projective space, and $(\mathbb{P}^2_{\mathbb{C}})^*$ parametrizes the pencil through $e_1 \in \mathbb{P}^2_{\mathbb{C}}$. The line $\overleftrightarrow{e_1e_2}^*$ is a point in $(\mathbb{P}^2_{\mathbb{C}})^*$.

In the dual space, the transformation g^* has the same normal form as g. That is, g^* is conjugated to g. Then, there is an invariant line in $(\mathbb{P}^2_{\mathbb{C}})^*$, $\overleftrightarrow{f_1 f_2}$, and a unique fixed point: f_1.

As $\overleftrightarrow{e_1 e_2}$ is invariant, $\overleftrightarrow{e_1 e_2}^*$ is a fixed point and therefore coincides with f_1. Recall that if $L = \overleftrightarrow{f_1 f_2}$, all points in the line L represent a line passing through e_1 in $\mathbb{P}^2_{\mathbb{C}}$.

Let ℓ be a line in $\mathbb{P}^2_{\mathbb{C}}$, such that e_1 and e_2 are not in ℓ; then ℓ^* represents a point in $(\mathbb{P}^2_{\mathbb{C}})^* \setminus L$. In this case, $(g_n^*)^*(\ell^*) = g_n(\ell) \longrightarrow f_1^* = \overleftrightarrow{e_1 e_2}$.

Given $y \in \overleftrightarrow{e_1 e_2}$, there is a sequence $(x_n) \subset \ell$ that converges to a point $x \in \ell$, such that $g_n(x_n) \to y$. Note that x is in $\overleftrightarrow{e_1 e_2}$; on the contrary, the compact set $(x_n) \cup x$ would be outside the line and would accumulate on e_1, according to the previous analysis. Therefore, $L_2(G) = \overleftrightarrow{e_1 e_2}$. □

Type (2.c)

The fixed points of the transformations are e_1 and e_3; however, there are more points with infinite isotropy group. For $(x, y, z) \in \mathbb{P}^2_{\mathbb{C}}$, we have $g^n(x, y, z) = (x + ny, y, \lambda^n z)$. In case a is rational, the entire line $\ell = \overleftrightarrow{[e_1 e_3]}$ is in $L_0(g)$.

The same transformation that was shown in (4) is obtained as the limit, if we consider the sequence of transformations generated by powers of g, divided each one by the entry of greater norm. So, if $a \in \mathbb{Q}$, and $(x, y, z) \notin \overleftrightarrow{[e_1 e_3]}$, $g^n(x, y, z)$ tends to e_1 iterating both, to the future and to the past. And $L_1(G) = \{e_1\}$.

When $a \in \mathbb{R} \setminus \mathbb{Q}$, the only points in L_0 are $\{e_1, e_3\}$. And for a point $(x, 0, z)$ in the line $\overleftrightarrow{[e_1 e_3]}$ is easy to build a sequence in $\mathbb{P}^2_{\mathbb{C}}$ converging to $(x, 0, z)$. It is enough to find a subsequence $\{n_k\}$, such that $\lambda^{n_k} \to 1$ when $n \to \infty$; as a is irrational, the subsequence exists. Then, $g^n(X) = (x, 0, \lambda^n z)$ converges to $(x, 0, z)$. Therefore, L_1 is the entire line.

Finally, for the compact subsets in $\mathbb{P}^2_{\mathbb{C}} \setminus L_0 \cup L_1$; in both cases, the union of these subsets is the line ℓ. And compact sets outside this line converge to e_1 as seen before.

3.3 Elliptic Transformations

An *elliptic* element in PSL(3, ℂ) are those whose lift in $SL(3, \mathbb{C})$ has the following Jordan canonical form:

$$g = \begin{bmatrix} e^{2\pi i \alpha} & 0 & 0 \\ 0 & e^{2\pi i \beta} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \beta \in \mathbb{R}. \tag{5}$$

Consider the elliptic element in equation (5). In the case that α and β are rational, the transformation has finite order, and therefore it has no fixed points nor accumulation points. Then, the limit set is empty.

If at least β is an irrational number, L_0 will be the fixed points of the transformation. For each (x, y, z) we can find a subsequence $\{x_n\}$, such that $g^n(x_n)$ converges to (x, y, z). For example, assuming that $\alpha = p/q$, we know that there are infinite natural numbers n, such that $e^{2\pi i n\alpha} = 1$. As β/q is still an irrational number, it is known that there is a subsequence n_k that makes the convergence of $e^{2\pi i\beta n_k}$ to 1. Therefore, if

$$x_n = e^{2\pi i n\alpha}, \quad y_n = e^{2\pi i n\beta}, \quad z_n = 1, \tag{6}$$

results that X is an accumulation point and $(x, y, z) \in L_1$. So the second set is $L_1 = \mathbb{P}^2_{\mathbb{C}}$, and $L_2 = \varnothing$.

4 Relation between $L(G)$ and $\Lambda(G)$ with G subgroup of $\mathbf{PU}(2,1)$

Recall that the sequence $(g_m)_{m\in\mathbb{N}} \subset \mathrm{PSL}(3,\mathbb{C})$ converges to $g \in QP(3,\mathbb{C})$ in the sense of quasi-projective transformations if $g_m \longrightarrow g$, when $m \to \infty$ uniformly on compact subsets of $\mathbb{P}^2_{\mathbb{C}} \setminus \ker(g)$.

We study the lemma 4.2 of [1].

Lemma 4.1. *Let G be a subgroup of $PU(2,1)$ a discrete subgroup, $(g_m)_{m\in\mathbb{N}} \subset G$ a sequence of distinct elements, and $g \in QP(3,\mathbb{C})\setminus PSL(3,\mathbb{C})$, such that $(g_m)_{m\in\mathbb{N}}$ converges to g in the sense of quasi-projective transformations. Then:*

(i) The image $i(g)$ is a point in $\partial\mathbf{H}^2_{\mathbb{C}}$.

(ii) $\ker(g)^{\perp}$ is a point in $\partial\mathbf{H}^2_{\mathbb{C}}$.

Proof. For the proof of (i), we use proposition 3.2 in [8], where it is asserted that given a sequence of distinct elements of a discrete subgroup $G \subset \mathrm{PU}(2,1)$, there is a subsequence and distinct elements $x, y \in L(G)$, such that $g_m(z) \to x$ uniformly in compact sets of $\overline{\mathbf{H}^2_{\mathbb{C}}} \setminus \{y\}$.

As $\mathbf{H}^2_{\mathbb{C}}$ is an open subset of $\mathbb{P}^2_{\mathbb{C}}$, g is a holomorphic transformation, then $g(\mathbf{H}^2_{\mathbb{C}})$ is an open subset in the image of g. On the other hand, $g(\mathbf{H}^2_{\mathbb{C}}) = p$ is a closed subset of $i(g)$. The only set which is closed and open at the same time is either the total set or the empty set, follows that $i(g) = \{p\}$.

Having said this, $i(g)$ is a point in the boundary of the hyperbolic space. To prove the second part of the lemma, recall that the sum of the dimension of the kernel of a transformation on a vector space plus the dimension of its image equals the sum of the vector space. If we consider that $g:\mathbb{C}^3 \longrightarrow \mathbb{C}^3$ is a linear transformation and that $dim_{\mathbb{C}}(i(g)) = 1$, we have $dim_{\mathbb{C}}(\ker(g)) = 2$, and this implies that $[\ker(g)]$ is a projective line.

Then, the claims are $\mathbf{H}^2_{\mathbb{C}} \cap \ker(g) = \varnothing$ and $\partial\mathbf{H}^2_{\mathbb{C}} \cap \ker(g) \neq \varnothing$. Both claims are proven by contradiction. In the first case, let $x \in \mathbf{H}^2_{\mathbb{C}} \cap \ker(g)$; as the transformation g is not identically zero, we can take $x \notin i(g)$. By proposition 3.3 in [1], for (g_m), g, x and $\mathbf{H}^2_{\mathbb{C}}$, follows that there is a line contained in $\mathbf{H}^2_{\mathbb{C}}$, which is a contradiction. Therefore, $\mathbf{H}^2_{\mathbb{C}} \cap \ker(g) = \varnothing$.

For the second claim, assume that $\partial\mathbf{H}^2_{\mathbb{C}} \cap \ker(g) = \varnothing$. By (i) of this same lemma, it exists $p \in \partial\mathbf{H}^2_{\mathbb{C}}$, such that g_m converges uniformly to p, which is a constant transformation. Let x be an element in $\mathbf{H}^2_{\mathbb{C}}$ and U a neighborhood of p that satisfies $U \cap \mathbf{H}^2_{\mathbb{C}} \subset \mathbf{H}^2_{\mathbb{C}} - \{x\}$. Then, there is $n_0 \in \mathbb{N}$, such that if $m > n_0$, $g_m(\mathbf{H}^2_{\mathbb{C}}) \subset U \cap \mathbf{H}^2_{\mathbb{C}} \subset \mathbf{H}^2_{\mathbb{C}} - \{x\}$. But this is a contradiction, given that g_m is a homeomorphism of $\mathbf{H}^2_{\mathbb{C}}$. □

λ-Lemma

The next result is known as the λ-Lemma. We'll use it in further arguments.

Lemma 4.2. *Let $g \in PU(2,1)$ be a loxodromic element with fixed points $a, r \in \mathbb{P}^2_{\mathbb{C}}$; and let $\Omega \subset \mathbb{P}^2_{\mathbb{C}}$ be an open subset. Assume that $\langle g \rangle$ acts properly discontinuously in Ω. Then, $a^{\perp} \in \mathbb{P}^2_{\mathbb{C}} \setminus \Omega$ or $r^{\perp} \in \mathbb{P}^2_{\mathbb{C}} \setminus \Omega$.*

Theorem 4.3. *The Kulkarni limit set coincide with the perpendicular lines tangent to $\partial\mathbf{H}^2_{\mathbb{C}}$ in points of the Chen-Greenberg limit set.*

Proof. In [5], Kamiya shows that a non-elementary discrete subgroup G, always has a loxodromic element g. By lemma 4.2, we have that $a^{\perp}$ belongs to the Kulkarni limit set. As we saw in section 3.1, also $r^{\perp}$ belongs to $\Lambda(G)$. Besides, $\Lambda(G)$ is an invariant set and the action of G is transitive.

The fixed points of loxodromic elements are dense in the Chen-Greenberg limit set [3], then all the tangent lines to the ball in points of $L(G)$ are, in fact, in $\Lambda(G)$. For transformations of type $(1.c)$, it happens that $\cup_{p\in L(G)}\ell_p \subset \Lambda(G)$.

To show the other contention, lemma 2.3 is applied; then, if $C = \cup_{p\in L(G)}\ell_p$, and we consider a compact set outside C, by lemma 4.1, this compact set acummulates on some point of $\partial\mathbf{H}^2_{\mathbb{C}}$. That is, $\Lambda(G) \subset \cup_{p\in L(G)}\ell_p$, and therefore

$$\Lambda(G) = \cup_{p\in L(G)}\ell_p. \tag{7}$$

□

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Veronese Groups *

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Abstract

In the following, we will describe the equicontinuous set of a family of discrete subgroups of $\mathrm{PSL}(3,\mathbb{C})(n+1,\mathbb{C})$ in aims to obtain a family of examples of complex Kleinian groups.

Keywords: Veronese groups, complex Kleinian groups, equicontinuous set.

1 Introduction

The theory of Kleinian groups was introduced by Poincaré in the decade of 1880, these groups are defined as monodromy groups of certain differential equation in the plane; these groups have a big role in different areas of mathematics as Riemann surfaces, Teichmüller spaces, automorphic forms, holomorphic dynamics, conformal and hyperbolic geometry.

Another way to understand these groups are: as subgroups of conformal transformations of the two sphere, as subgroups of $\mathrm{PSL}(2,\mathbb{C})$ acting on the projective line and as subgroups of the isometry group of the hyperbolic plane. The three ways to understand have different properties and these ones varyes by the geometry of the space where the groups act. These three ways of work of the Kleinian groups are have enriched the classical Kleinian group theory, we can mention [2]. We recall that a Kleinian group gives a partition of the space where is acting, in two invariant sets: one is the "limit set", that is, the minimal closed set where occurs interesting dynamical features, for example in some cases is a perfect set; the other is the complement of the first one and it's called the "domain of discontinuity", the maximal open set where the action is discontinuous.

A big part of the Kleinian group theory has been generalized to higher dimensional setting, as subgroups of conformal transformations of the $n-$dimensional sphere that have a non-empty open set where its action is discontinuous; these groups are known as *conformal Kleinian groups* (see [12]).

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In [15] was introduced the notion of a *complex Kleinain group* as discrete subgroups of $\mathrm{PSL}(n+1,\mathbb{C})$ acting on the $n-$dimensional projective space and that have a non-empty open subset where the action of the group is properly discontinuous. After, in [13] and [14], was estudied a notion of *limit set* (and its complement) for a special family of complex Kleinian groups in $\mathrm{PSL}(3,\mathbb{C})$, known as *complex hyperbolic groups.* Later, in [4], was described the complement of the *equicontinuous set* for higher dimensional complex hyperbolic groups, work that generalice the ideas in the two dimensional setting. After all the work done for complex Kleinian groups, there is still a lot of empty spaces to hard work on. In this paper, we studied the equicontinuous set and its complement for an special kind of subgroups, that we call *Veronese groups*, and this in the aims of look for a different example of Kleinian groups in higher dimensional setting.

2 Preliminaries

Let $\mathbb{C}\mathrm{P}^n$ be the $n-$*dimensional complex projective space*, that is, the set of conjugacy classes of $\mathbb{C}^{n+1}\setminus\{0\}$ under the natural action of $\mathbb{C}^*$. The *projective special group*, denoted by $\mathrm{PSL}(n+1,\mathbb{C})$, is the set of conjugacy classes of the special lineal group $\mathrm{SL}(n+1,\mathbb{C})$ under the natural action of $\mathbb{C}^*$. Note that $\mathrm{PSL}(n+1,\mathbb{C})$ has a natural action on $\mathbb{C}\mathrm{P}^n$ at the level of representatives of conjugacy classes, *i.e.*,

$$[g][z]=[gz],\ \ [g]\in \mathrm{PSL}(n+1,\mathbb{C}),\ \text{and}\ [z]\in\mathbb{C}\mathrm{P}^n.$$

The elements of $\mathrm{PSL}(n+1,\mathbb{C})$ are classified in terms of its linear properties.

Definition 2.1. An element $\gamma\in\mathrm{PSL}(n+1,\mathbb{C})$ it is said to be:

- *elliptic,* if and only if has a lift $\tilde{\gamma}\in\mathrm{SL}(n+1,\mathbb{C})$, that is, diagonalizable and all of its proper values are unitary complex numbers.
- *parabolic*, if every lift $\tilde{\gamma}\in\mathrm{SL}(n+1,\mathbb{C})$, that is, non-diagonalizable and each of its proper values is are unitary complex numbers.
- *loxodromic*, if and only if has a lift $\tilde{\gamma}\in\mathrm{SL}(n+1,\mathbb{C})$ with at least one non-unitary proper value. If every proper value of $\tilde{\gamma}$ are different and it's diagonalizable, then we will say that γ is *purely strongly loxodromic.*

Remark 2.2. The previous Definition generalize the trichotomy of the elements of $\mathrm{PSL}(2,\mathbb{C})$; for more information of the dynamical features for the elements described previously, see [7].

A *complex Kleinian group* is a discrete subgroup G of $\mathrm{PSL}(n+1,\mathbb{C})$, such that there exist an open subset of $\mathbb{C}\mathrm{P}^n$ where the action of G is properly discontinuous (see [3]). In [13], [6] they studied some examples of complex Kleinian groups as discrete subgroups of $\mathrm{PU}(n,1)$, these groups has an invariant sphere in $\mathbb{CP}^n$, that is, the model of the *complex hyperbolic space* (see [10]). Another example was given in [5], as Schottky subgroups of $\mathrm{PSL}(n+1,\mathbb{C})$ in $\mathbb{CP}^n$. Given a discrete subgroup Γ of $\mathrm{PSL}(n+1,\mathbb{C})$, the *limit set* of Γ is the closure of the cluster points of the orbits of Γ of points in $\mathbb{CP}^n$ and we will denoted it by $\Lambda(\Gamma)$.

Remark 2.3. Maximal and minimal properties on the discontinuity set and the limit set are classical features for Kleinian groups. We recall that for discrete subgroups of $\mathrm{PSL}(n+1,\mathbb{C})$, the limit set described above doesn't imply that its complement is the maximal open subset of $\mathbb{CP}^n$ where the action is properly discontinuous, see [13] for an example that this doesn't occurs. In [1] its proved that for a generic family of discrete subgroups there exists a unique way to obtain a minimal closed invariant subset.

In our, we will use the limit set defined above to construct a new closed invariant subset for our groups; even if the new closed subset isn't minimal, its complement is an open subset where the action of the group is properly discontinuous.

The following tool was introduced on [4] for the study of the equicontinuous set of discrete subgroups of $\mathrm{PU}(n,1)$.

Let $T:\mathbb{C}^{n+1}\to\mathbb{C}^{n+1}$ be a linear transformation with non-trivial kernel and denote $[\ker T]$ the projectivization of its kernel; the map T induce a map from $\mathbb{CP}^n\setminus[\ker T]$ to $\mathbb{CP}^n$, denoted by $[\![T]\!]$, given by

$$[\![T]\!]([z])=[T(z)]. \tag{1}$$

We will call the map $[\![T]\!]$ a *pseudo-projective map* and if $M(n+1,\mathbb{C})$ denotes the set of linear transformations from $\mathbb{C}^{n+1}$ to $\mathbb{C}^{n+1}$, the set $M(n+1,\mathbb{C})\setminus\{0\}/\mathbb{C}^*$ induces the set of pseudo-projective maps and we will denoted it by $\mathrm{PsP}(n+1,\mathbb{C})$.

The following proposition relates the pointwise convergence in $\mathrm{PsP}(n+1,\mathbb{C})$ and uniform convergence in $\mathbb{CP}^n$.

Proposition 2.4. *(see [4]) Let $(\gamma_m)_{m\in\mathbb{N}}\subset\mathrm{PSL}(n+1,\mathbb{C})$ be a sequence of distinct elements, then:*

1. *There is a subsequence $(\gamma_{m_j})_{j\in\mathbb{N}}$ and $\gamma_0\in\mathrm{PsP}(n+1,\mathbb{C})$, such that $\gamma_{m_j}\xrightarrow[j\to\infty]{}\gamma_0$ as points in $\mathrm{PsP}(n+1,\mathbb{C})$.*
2. *If $(\gamma_{m_j})_{j\in\mathbb{N}}$ is the sequence given by the previous part of this lemma, then $\gamma_{m_j}\xrightarrow[j\to\infty]{}\gamma_0$, as functions, is uniformly on compact sets of $\mathbb{CP}^n\setminus[\ker\gamma_0]$.*

Let G be a group acting on a manifold X. A point z in X is said to be *equicontinuous* if there is an open neighbourhood U of z, such that $G|_U$ is a normal family, *i.e.,* every sequence of distinct elements has a subsequence that converges uniformly on compact sets relative to U. The *equicontinuity region* of G, denoted by $\mathrm{Eq}(G)$, is the set of equicontinuous points $z \in X$. The Proposition 2.4 implies that for a sequence of different elements in $\mathrm{PSL}(n+1, \mathbb{C})$, the equicontinuous set of the sequence is given by the complement in $\mathbb{C}\mathrm{P}^n$ of the kernel of the limit pseudo-projective transformation (see [4]).

3 The Irreducible Representation and the Veronese curve

Let H_n be the vector space of homogeneous complex polynomials in two variables of degree n with the natural basis $\beta = e_j(z,w) = z^{n-j}w^j{}_{j=0}^{n}$. We will denote by $\mathbb{P}(H_n)$ the set of classes under the natural action of $\mathbb{C}^*$ in $H_n \setminus \{0\}$.

The projective special linear group $\mathrm{PSL}(2,\mathbb{C})$ has a natural action on $\mathbb{P}(H_n)$ as a change of variables in a polynomio of a class:

$$\begin{array}{rccc} \rho: & \mathrm{PSL}(2,\mathbb{C}) \times \mathbb{P}(H_n) & \to & \mathbb{P}(H_n) \\ & ([A],[p(z,w)]) & \mapsto & [p\left((z,w)\cdot A\right)] \end{array} \tag{2}$$

Remark 3.1. There are morphisms that identify $\mathbb{P}(H_n)$ with the $n-$symmetric power of $\mathbb{C}\mathrm{P}^1$, to know $(\mathbb{C}\mathrm{P}^1)^n/S_n$, and this one with $\mathbb{C}\mathrm{P}^n$. So we can think that the ρ action is a $\mathrm{PSL}(2,\mathbb{C})$ action on $\mathbb{C}\mathrm{P}^n$.

We can translate the ρ action to a matrix representation, if we look in how a matrix A acts on the elements of the basis $\{e_j(z,w)\}$, computing:

$$\rho(A,[e_m]) = (az+cw)^{n-m}(bz+dw)^m \quad \text{with} \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

With a straigth computation, we can deduce:

$$\rho(A,[e_m(z,w)]) = \sum_{j=0}^{n} \left[\sum_{k=\delta_{j,m}}^{\Delta_{j,m}} \binom{n-m}{k}\binom{m}{j-k} a^{n-m-k}c^k b^{m-j+k} d^{j-k} \right] z^{n-j}w^j, \tag{3}$$

where $\delta_{j,m} = \max\{j-m, n\}$ and $\Delta_{j,m} = \min\{j, n-m\}$, and we can do this for every $m = 0, \cdots, n$. So we obtain a application between $\mathrm{PSL}(2,\mathbb{C})$ and $\mathrm{PSL}(n+1,\mathbb{C})$, that we will still denote ρ.

Lemma 3.2. *The map* $\rho : \mathrm{PSL}(2,\mathbb{C}) \to \mathrm{PSL}(n+1,\mathbb{C})$ *described above, is a well defined group morphism; even more, it is injective.*

Proof. The group morphism property follows from the action ρ. The injectivity follows from the form of $\rho([A])$ for a class $[A]$. □

The morphism ρ is known as the *irreducible representation* of $\mathrm{PSL}(2,\mathbb{C})$ into $\mathrm{PSL}(n+1,\mathbb{C})$, see [11]. In the following, we will describe some properties related to the morphism and the elements of $\mathrm{PSL}(\cdot,\mathbb{C})$.

Proposition 3.3. *The morphism* ρ *is type preserving,* i.e., *it sends loxodromic, parabolic, and elliptic elements of* $\mathrm{PSL}(2,\mathbb{C})$ *into loxodromic, parabolic, and elliptic elements of* $\mathrm{PSL}(n+1,\mathbb{C})$. *Even more, if* Γ *is a discrete subgroup of* $\mathrm{PSL}(2,\mathbb{C})$, *then* $\rho(\Gamma)$ *is a discrete subgroup of* $\mathrm{PSL}(n+1,\mathbb{C})$.

Proof. For the part of type preserving, it will be sufficient to look at the image of the matrices

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

a straight computation involving Equation (3) implies the claim.

For the second part of the proposition, suppose that $\Gamma < \mathrm{PSL}(2,\mathbb{C})$ is a discrete subgroup, but $\rho(\Gamma)$ isn't discrete. So there is a sequence $(A_m)_{m\in\mathbb{N}}$ in Γ, such that $\rho(A_m) \to Id_{n+1}$ as $m \to \infty$, with

$$A_m = \begin{bmatrix} a_m & b_m \\ c_m & d_m \end{bmatrix}.$$

From the expression of $\rho(A_m)$, we can imply that $A_m \to Id_2$; this is a contradiction because Γ is discrete. Therefore, $\rho(\Gamma)$ is discrete. □

There is a way to embed $\mathbb{CP}^1$ into a curve in $\mathbb{CP}^n$ and it comes from the construction of the morphism ρ. Let be,

$$\begin{array}{rccc} \psi : & \mathbb{CP}^1 & \to & \mathbb{CP}^n \\ & [z:w] & \mapsto & \left[z^n : \cdots : \binom{n}{m} z^{n-m} w^m : \cdots : w^n\right] \end{array} \tag{4}$$

this map is known as the *Veronese embedding.* We will denote by $\mathscr{C}_n$ the image of $\mathbb{CP}^1$ under ψ; the set $\mathscr{C}_n$ is a normal rational algebraic curve called the *Veronese curve.*

The following Lemma involve geometric properties of the Veronese curve:

Lemma 3.4. *Every subset of* $n+1$ *different points in* $\mathscr{C}_n$ *is linearly independent.*

Proof. Let $\{p_j = \psi([1 : t_j])\}_{j=0}^{n}$ be with $t_j \neq t_k$ if $j \neq k$. Let $\sum_{j=0}^{n} a_j p_j = 0$ be a linear combination; it follows from the form of the elements that the linear system with variables a_j's has the unique solution 0. Therefore, $\{p_j\}_{j=0}^{n}$ is linearly independent. □

Proposition 3.5. *Every subset of $m > n+1$ different points of $\mathscr{C}_n$ is in general position.*

Proof. The proof follows from the previous lemma. □

Proposition 3.6. *The group of projective automorphisms of $\mathscr{C}_n$ is $\rho(\mathrm{PSL}(2,\mathbb{C}))$.*

Proof. A straight computation give us that $\rho(A)\cdot\psi(p) = \psi(A\cdot p) \in \mathscr{C}$ for every $A \in \mathrm{PSL}(2,\mathbb{C})$ and $p \in \mathbb{C}\mathrm{P}^1$. We can conclude that $\rho(\mathrm{PSL}(2,\mathbb{C}))$ leaves invariant the curve $\mathscr{C}$. Let us suppose that there is an element $B \in \mathrm{PSL}(n+1,\mathbb{C})$, such that $B(\mathscr{C}) = \mathscr{C}$. Denote by $\tilde{B}$ the map from $\mathbb{C}\mathrm{P}^1$ into $\mathbb{C}\mathrm{P}^1$ given by $\tilde{B}([z:w]) = \psi^{-1}B\psi([z:w])$. The map $\tilde{B}$ is an holomorphic map, so belongs to $\mathrm{PSL}(2,\mathbb{C})$; even more, the following diagram conmutes:

$$\begin{array}{ccc} \mathbb{C}\mathrm{P}^1 & \xrightarrow{\tilde{B}} & \mathbb{C}\mathrm{P}^1 \\ \psi\downarrow & & \downarrow\psi \\ \mathscr{C} & \xrightarrow[B]{} & \mathscr{C} \end{array}.$$

We can assure that $B|_{\mathscr{C}} = \rho(\tilde{B})|_{\mathscr{C}}$. Let us take $n+2$ different points of $\mathscr{C}$ in general position, the transformation $B\rho(\tilde{B})^{-1}$ fixes the $n+2$ points; this implies that $B\rho(\tilde{B})^{-1} = Id_{n+1}$ in $\mathbb{C}\mathrm{P}^n$. So, the projective automorphisms of $\mathscr{C}$ is $\rho(\mathrm{PSL}(2,\mathbb{C}))$. □

Corollary 3.7. *The Veronese embedding ψ is $\mathrm{PSL}(2,\mathbb{C})-$equivariant. Even more, for every $\Gamma \subset \mathrm{PSL}(2,\mathbb{C})$ group, ψ is $\Gamma-$equivariant.*

Remark 3.8. The previous Corollary implies that the action of subgroups of $\rho(\mathrm{PSL}(2,\mathbb{C}))$ on $\mathscr{C}$ is essentially the well-known action of subgroups of $\mathrm{PSL}(2,\mathbb{C})$ in $\mathbb{C}\mathrm{P}^1$. Even more, if $z \in \mathbb{C}\mathrm{P}^1$ belongs to the limit set of Γ, then $\psi(z)$ is an accumulation point for the orbits of $\rho(\Gamma)$.

Corollary 3.9. *Let Γ be a discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$, and $\Lambda(\Gamma)$ its limit set. Then $\psi(\Lambda(\Gamma))$ is contained in $\Lambda(\rho(\Gamma))$.*

Definition 3.10. A Veronese group is the image under ρ of a discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$.

Definition 3.11 ([11]). Let $\phi : \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^n$ be a smooth curve defined by

$$\phi([z:w]) = [v_0(z,w) : \cdots : v_n(z,w)] .$$

The osculating hyperplane of $\phi(\mathbb{C}\mathrm{P}^1)$ at $p = \phi([1:k])$ is the row space of the matrix

$$\left.\begin{pmatrix} v_0 & v_1 & \cdots & v_{n-1} & v_n \\ v_0^{(1)} & v_1^{(1)} & \cdots & v_{n-1}^{(1)} & v_n^{(1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ v_0^{(n-1)} & v_1^{(n-1)} & \cdots & v_{n-1}^{(n-1)} & v_n^{(n-1)} \end{pmatrix}\right|_{[1:k]}. \tag{5}$$

Remark 3.12. We recall that

i. The osculating hyperplane of a curve $\phi(\mathbb{C}\mathrm{P}^1)$ is an hyperplane in $\mathbb{C}\mathrm{P}^n$ that intersects the curve in just one point.

ii. There is a mapping $\Pi_n : \phi(\mathbb{C}\mathrm{P}^1) \to (\mathbb{C}\mathrm{P}^n)^*$, where $(\mathbb{C}\mathrm{P}^n)^*$ is the dual space of $\mathbb{C}\mathrm{P}^n$ that identifies a point with its osculating hyperplane and it's known as a polarity; we refer to [9] and [8] for more information.

Lemma 3.13. *The osculating hyperplane of $\mathscr{C}$ at $p = \psi([1:t])$ is given by the equation*

$$\mathscr{L}_p := \sum_{j=0}^{n} (-1)^j t^j z_{n-j} = 0, \tag{6}$$

where $[z_0 : \cdots : z_n]$ are the homogeneous coordinates of $\mathbb{C}\mathrm{P}^n$.

Proof. A parametrization for the Veronese curve is given by $v_j([1:t]) = \binom{n}{j} t^j$; a straight computation implies the claim. □

4 The Equicontinuity Region

The equicontinuity region of a group is an interesting set that gives relevant information of the dynamics of the group. The following theorem relates this set and the classical limit set of a Kleinian group:

Theorem 4.1. *Let Γ be a discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$, and $G = \rho(\Gamma)$ the correspondient Veronese group, then*

$$\mathbb{C}\mathrm{P}^n \setminus \mathrm{Eq}(G) = \bigcup_{z \in \Lambda(\Gamma)} T_{\psi(z)} \mathscr{C}, \tag{7}$$

where $\Lambda(\Gamma)$ is the limit set of Γ for its action on $\mathbb{C}\mathrm{P}^1$, and $T_z\mathscr{C}$ is the osculating hyperplane to $\mathscr{C}$ in z.

Proof. We can extend by continuity the morphism ρ to a map from $\mathrm{PsP}(2,\mathbb{C})$ into $\mathrm{PsP}(n+1,\mathbb{C})$, if $\gamma = \lim_{m\to\infty} \gamma_m \in \mathrm{PsP}(2,\mathbb{C})$, then $\rho(\gamma) = \lim_{m\to\infty} \rho(\gamma)$.

Let us assume that after a conjugation $[1:0],[0:1]$ doesn't belong to $\Lambda(\Gamma)$. Let $[1:k] \in \Lambda(\Gamma)$, following the ideas of [4], there is a sequence $(\gamma_n)_{n\in\mathbb{N}} \subset \Gamma$, such that $\gamma_n \to \gamma \in \mathrm{PsP}(2,\mathbb{C})$ and $\ker\gamma = [1:k]$; even more, we can assure that

$$\gamma = \begin{bmatrix} -bk & b \\ -dk & d \end{bmatrix}$$

and by Corollary 3.9 and the previous paragraph, under the map ρ we have a sequence in $\mathrm{PSL}(n+1,\mathbb{C})$ that converges to an element in $\mathrm{PsP}(n+1,\mathbb{C})$; by the continuity, we have that this limit is $\rho(\gamma)$. A straight computation gives us that the kernel of $\rho(\gamma)$ is spanned by $\beta = \{e_1 - (-k)^m e_{m+1}\}$, where $\{e_i\}_{i=1}^{m+1}$ is the standard basis of $\mathbb{C}^{n+1}$. By lemma 3.13, the hyperplane spanned by β is the osculating hyperplane of $\mathscr{C}$ at $\psi([1:k])$, and by proposition 2.4,
we obtain the theorem. ☐

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Geometry of AdS^n *

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Abstract

Anti de Sitter space, abbreviated AdS^n, is a lorentzian model space, and an example cosmological model. It is the analogous of hyperbolic space, $\mathbb{H}^n$, and has some applications in physics. In these notes we review the geometrical properties of AdS^n, their analogy with classical results from real hyperbolic geometry and show that there is a relation between AdS^3 and group theory.

Keywords: Hyperbolic geometry, anti de Sitter space, differential geometry.

1 Introduction

Anti de Sitter space is one of the spacetime models, collectively known as Robertson-Walker spaces in the physics literature [5]. In mathematical terms, anti de Sitter space is an homogeneous semi-Riemannian manifold of constant negative sectional curvature, and, as we shall see, it is the lorentzian analogous of real hyperbolic space. In recent years, it has gained relevance in the physics literature, after the AdS/CFT correspondence, developed by string theorists, has proved to be a powerful tool in studying field theory phenomena with tools from gravity physics of a more geometrical nature [7]. This article collects the results from anti de Sitter spacetime that can be found in several places, and that I presented in the workshop on Kleinian groups and related topics, held in the Institute of Mathematics of the National Autonomous University of Mexico, at Cuernavaca city.

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2 Some hyperbolic geometry

Let $\mathbb{R}^{n,1}$ denote $\mathbb{R}^{n+1}$ equipped with the quadratic form

$$\eta = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}.$$

Recall that *hyperbolic space*, $\mathbb{H}^n$, is the upper half space

$$\mathbb{H}^n = \{x \in \mathbb{R}^n : x_n > 0\}$$

endowed with the Riemannian metric:

$$\frac{dx_1^2 + \cdots + dx_n^2}{x_n^2}.$$

There are three custom isommetric models of hyperbolic space. The other two models are the *Poincaré disc*,

$$\mathbb{D}^n = \left\{x \in R^n : x_1^2 + \cdots + x_n^2 < 1\right\}$$

with the metric

$$4\,\frac{dx_1^2 + \cdots + dx_n^2}{(1 - ||x||^2)^2},$$

and the *hyperboloid model*,

$$\left\{x \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_n^2 = x_{n+1}^2 - 1,\, x_{n+1} > 0\right\}$$

with the subspace metric inherited from the quadratic form η defined above.

Note that the hyperboloid model corresponds to the set of solutions to $x^t \eta x = -1$, restricted to the upper halfspace $x_{n+1} > 0$.

It is customary to use the three models according to the context interchangeably, since some results are easier to work in one but not the others. See [2] for a survey of the models and other important concepts in the field.

Theorem 2.1. *Let $x', x \in \mathbb{H}^n$ be two points in the hyperboloid model. Let d be the hyperbolic distance between the points. Then*

$$\cosh(d) = -x^t \eta x'.$$

Compare with [1], page 6.

Proof. Denote by x^2 the product $x^t\eta x$. Consider the Lagrangian $L = \frac{1}{2}\dot{x}^2 + \lambda(x^2+1)$, and the corresponding functional

$$\int L(x,\dot{x})dt.$$

L is an energy Lagrangian with a corresponding undetermined coefficient restricting solutions to the hyperboloid model. By working out Euler-Lagrange equations, it turns out that extremal solutions must obey the system of equations

$$\ddot{x}_i = \lambda x_i, \qquad\qquad x^2 = -1.$$

Note that, since $x^t\eta x = -1$, the velocity field in $\mathbb{R}^{(n,1)}$ must obey $x^t\eta\dot{x} = 0$, and differentiating once again, it turns out that

$$\dot{x}^2 = \lambda.$$

Therefore, $\lambda > 0$, and the solutions for x are of the form $x_i = A_i e^{\sqrt{\lambda}\tau} + B_i e^{-\sqrt{\lambda}\tau}$. Let $A = (A_i,\ldots,A_{n+1})$ and $B = (B_1,\ldots,B_{n+1})$. Since $x^2 = -1$, it follows that

$$A^2 e^{\sqrt{\lambda}\tau} + 2A^t\eta B + B^2 e^{-\sqrt{\lambda}\tau} = -1$$

for any τ. Therefore,

$$A^2 = 0, \qquad B^2 = 0, \qquad 2A^t\eta B = -1.$$

Suppose that x' and x are points on the minimizing geodesic so described, say, $x = x(\tau_1)$, $x' = x(\tau_2)$. Then,

$$x^t\eta x' = x(\tau_1)^t\eta x(\tau_2) = A^t\eta B\left(e^{\sqrt{\lambda}(\tau_1-\tau_2)} + e^{-\sqrt{\lambda}(\tau_1-\tau 2)}\right).$$

The result follows in this case, since in the right hand side, $\sqrt{\lambda}(\tau_1-\tau_2)$ corresponds to the geodesic distance between the points. Upon substituting the expression for $A^t\eta B$, it follows that the right hand side is $-\cosh(d)$ in this case. The general result follows, since the hyperbolic model is symmetric and totally geodesic. □

Proposition 2.2. *The antipodal map, $-I$ acts by isometries on $\mathbb{R}^{n,1}$, and therefore, the lower hyperboloid*

$$\left\{x \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_n^2 = x_{n+1}^2 - 1,\, x_{n+1} < 0\right\}$$

is isometric to the upper.

Observe that in the hyperboloid model, isometries can be easily described.

Corollary 2.3. *Hyperbolic space is isometric to*

$$\left\{x \in \mathbb{R}^{n+1} : x^t \eta x = -1\right\} / \{\pm I\}$$

with the metric induced by the projection from the ambient $\mathbb{R}^{n,1}$ *metric.*

Proposition 2.4. *The isometries of* $\mathbb{H}^n$ *are the isometries of* η*, up to a factor* $\pm I$*,*

$$\operatorname{Isom}\left(\mathbb{H}^n\right) = \mathrm{PO}(n,1) = \left\{A \in \mathrm{GL}(n+1,\mathbb{R}) : A^t \eta A = \eta\right\} / \{\pm I\}.$$

And the orientation preserving isometries are the isometries lying in the identity component of $\mathrm{PO}(n,1)$*:*

$$\operatorname{Isom}^+\left(\mathbb{H}^n\right) = \mathrm{PO}_0(n,1).$$

Proofs can be found in [4], *e.g.,* section 3.114. For the general case, see [8], theorem 2.4.4.

Theorem 2.5. *For n odd,* $\mathrm{PO}(n,1) \cong \mathrm{SO}(n,1)$*, and* $\mathrm{PO}_0(n,1) \cong \mathrm{SO}_0(n,1)$*. For n even,* $\mathrm{PO}_0(n,1) \cong \mathrm{PSO}(n,1)$*.*

Proof. If n is odd and $A \in \mathrm{PO}(n,1)$, then $\det A = \det(-A)$. The claim follows. □

Remark 2.6. In the hyperboloid model, one usually describes $\mathrm{PO}(n,1)$ as the subgroup of $\mathrm{O}(n,1)$ preserving the positive sheet. In relativity, this subgroup is usually called the *Lorentz orthochronous* group.

Proposition 2.7. $\mathbb{H}^n$ *is homogeneous, isotropic, and symmetric. The isotropy group of* e_{n+1} *in the hyperboloid model is isomorphic to* $O(n)$*.*

Corollary 2.8. $\mathbb{H}^n \cong \mathrm{PO}(n,1)/\mathrm{O}(n)$*.*

Proof. From the previous proposition, we know the isometry and isotropy groups. Since $\mathbb{H}^n$ is simply connected, the result follows from the Theory of homogeneous spaces. □

2.1 The projective model

Note that in the Poincaré disc model and halfspace model, it is obvious the existence of a limiting boundary at infinity. In some sense, the projective model complements the hyperboloid model, and makes rigorous the common sense, that the null cone, should be the *boundary at infinity* of the hyperboloid.

Definition 2.9. The projective model is the domain in $\mathbb{R}P^n$ given by

$$\left\{x \in \mathbb{R}^{n+1} : x^t \eta x < 0\right\} / scale,$$

where *scale* refers to the equivalence relation given by $x \sim \lambda x$, $\lambda \in \mathbb{R}\backslash\{0\}$.

The projective model is also called *Klein's model* [2].

Proposition 2.10. *The application $[x] \mapsto x/\sqrt{-x^2}$ defines an isomorphism from the projective model to the hyperboloid model of $\mathbb{H}^n$.*

By pulling back the hyperbolic metric, we obtain the metric in the projective model. Note that, in the $x_{n+1} = 1$ patch for $\mathbb{R}P^n$, with coordinates $[x_1 : \ldots : x_n : 1]$, the projective model is the disc $x_1^2 + \cdots + x_n^2 < 1$.

Theorem 2.11. *In projective coordinates $[x_1 : \ldots : x_n : 1]$, the hyperbolic metric takes the form*

$$\frac{\sum dx_1^2}{1 - \sum x_i^2} + \frac{\left(\sum x_i dx_i\right)^2}{\left(1 - \sum x_i^2\right)^2}.$$

Proof. A straightforward calculation shows this. □

Definition 2.12. The ideal boundary at infinity, $\partial^\infty \mathbb{H}^n$, is the boundary of the closure of $\mathbb{H}^n$ in $\mathbb{R}P^n$.

Given the previous definitions, we state the next theorem:

Theorem 2.13. *$\partial^\infty \mathbb{H}^n$ is the set $\left\{x \in \mathbb{R}^{n+1} : x^t \eta x = 0\right\} / scale$, and corresponds to the limiting sphere $\sum_{i=1}^n x_i^2 = 1$ in the projective patch $[x_1 : \ldots : x_n : 1]$.*

Proof. It is immediate from the definition. □

Note that the ideal boundary at infinity corresponds to the projectivization of the null cone in Minkowski space. Using $\partial^\infty \mathbb{H}^n$, we can extend a well-known fact from the Poincaré and Half space models: Any geodesic in $\mathbb{H}^n$ is determined by its end points in the boundary at infinity.

Theorem 2.14. *Geodesics in $\mathbb{H}^n$ topologically correspond to the intersection of projective lines with the projective model.*

Proof. Recall from the hyperboloid model that geodesics in $\mathbb{H}^n$ correspond to hyperbolas in the paraboloid, parametrized by $Ae^{\sqrt{\lambda}t} + Be^{-\sqrt{\lambda}t}$, with A, B null vectors. Upon projectivization, these geodesics correspond to projective lines, whose intersection with $\partial^\infty\mathbb{H}^n$ are $[A]$ and $[B]$. □

Corollary 2.15. *A geodesic in $\mathbb{H}^n$ is determined by two distinct endpoints on $\partial^\infty\mathbb{H}^n$.*

3 AdS^n Space

In this section, η will denote de quadratic form in $\mathbb{R}^{n-1,2}$ given in matrix form as

$$\begin{pmatrix} I_{n-1} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The order of presentation is based in [3]. In some cases, I added proofs of my own, or that can be found elsewhere.

Definition 3.1.

$$\mathrm{AdS}^n = \left\{x \in \mathbb{R}^{n-1,2} : x^t\eta x = -1\right\} / \{\pm I\}$$

with semi-Riemannian metric given by restricting the ambient metric.

In coordinates, before the quotient, anti de Sitter space is the locus of points in Minkowski space, satisfying the equation $x_1^2 + \cdots + x_{n-1}^2 - x_n^2 - x_{n+1}^2 = -1$. Therefore, it is a one sheeted hyperboloid. However, unlike hyperbolic space, AdS^n is a *Lorentzian manifold*. The general Theory of semi-Riemannian manifolds can be found in [6].

Proposition 3.2. *The restriction of the metric in $\mathbb{R}^{n-1,2}$ to* AdS^n *has signature $(n-1, 1)$.*

Proof. Since $x^2 = -1$, the tangent space to AdS^n in x is given by the equation

$$x_1y_1 + \cdots + x_{n-1}y_{n-1} - x_ny_n - x_{n+1}y_{n+1} = 0.$$

i.e., the tangent space is orthogonal to x. As the metric is non degenerated and x is timelike, the restriction to the tangent space must have lorentzian signature. □

Note that the construction of AdS^n is analogous to the construction of the hyperboloid model. We will exploit this analogy very often.

Definition 3.3. The *causality* of a vector in a semi-Riemannian space is given by the sign of $||x||^2$, the corresponding semi-Riemannian norm, and can be one of the following:

$$\begin{cases} \text{spacelike}, & \text{if } ||x||^2 > 0, \\ \text{lightlike or null}, & \text{if } ||x||^2 = 0, \\ \text{timelike}, & \text{if } ||x||^2 < 0. \end{cases}$$

The causality of a curve is the causality of its velocity vectors, provided all have the same.

Remark 3.4. There is a notion in semi-Riemannian geometry of *causal curves*, which are curves whose velocity vectors are either timelike of null, but can't be spacelike.

Remark 3.5. Since geodesics are curves of constant length, they must be curves of the same causality at every point.

Theorem 3.6. *Geodesics $x(\tau)$ in AdS^n must be projections of solutions to $\ddot{x} = \lambda x$ in $\mathbb{R}^{n-1,2}$, restricted to $x^2 = -1$ and $\dot{x}^2 = \lambda$.*

Proof. The proof mimics the one given for the hyperboloid model. Let $L = \frac{1}{2}\dot{x}^2 + \frac{\lambda}{2}\left(x^2+1\right)$ be the energy Lagrangian with a Lagrange multiplier restricting solutions to AdS^n. Applying Euler-Lagrange equations, we obtain the conditions on the second derivative and $x(\tau)$. Derive twice the expression $x^2 = -1$, and recall that x^2 is a shorthand notation for $x^t\eta x$. Then,

$$2\dot{x}^2 + 2x^t\eta\ddot{x} = 0.$$

The condition on $\dot{x}$ follows, substituting $\ddot{x} = \lambda x$, and $x^2 = -1$. □

Solving the equation for $\ddot{x}$ and using the conditions for $\dot{x}$ and x, we can have an explicit description of the geodesics in anti de Sitter space.

Corollary 3.7. *Geodesics in AdS^n are given by:*

$$\begin{aligned} \text{Spacelike}: \quad & x(\tau) = Ae^{\sqrt{\lambda}\tau} + Be^{-\sqrt{\lambda}\tau}, & A^2 = B^2 = 0, \quad 2AB = -1, \\ \text{Null}: \quad & x(\tau) = A\tau + B, & A^2 = 0, \quad B^2 = -1, \quad AB = 0, \\ \text{Timelike}: \quad & x(\tau) = A\cos(\sqrt{-\lambda}\tau) + B\sin(\sqrt{-\lambda}\tau), & A^2 = B^2 = -1, \quad AB = 0, \end{aligned}$$

where AB stands for $A^t\eta B$.

Remark 3.8. In rigor, we have just proved the expression for geodesics given the initial data A, B, satisfying the conditions in the corollary.

Corollary 3.9. *Let $x \in \mathrm{AdS}^n$ and $p \in T_x\mathrm{AdS}^n$, then there exist constant points $A, B \in \mathbb{R}^{n-1,2}$, such that the local expression for the geodesic passing through x at velocity p is one of those given in the previous corollary.*

Proof. Since $\mathbb{R}^{n-1,2}$ is flat, we can identify p with a point in ambient space satisfying $x^t\eta p = 0$. Let $\lambda = ||p||^2$. Define A and B according to the causality of p:

$$\begin{array}{lll} \lambda > 0: & A = \dfrac{x + p/\sqrt{\lambda}}{2}, & B = \dfrac{x - p/\sqrt{\lambda}}{2}, \\ \lambda = 0: & A = p, & B = x, \\ \lambda < 0: & A = x, & B = p/\sqrt{-\lambda}. \end{array}$$

□

Remark 3.10. Unlike the hyperboloid model, in anti de Sitter space it is no longer true that any two points can be joined by a geodesic. The causal structure, which must be preserved, imposes restrictions for AdS^n to be geodesically connected as can be seen from the expressions in the previous corollary.

Lemma 3.11. *If ψ, ϕ are isometries of a connected, semi-Riemannian manifold X, such that for some $p \in X$, $\phi(p) = \psi(p)$, and $d\phi_p = d\psi_p$, then both isometries are the same.*

Proof. Since X is connected, any two points can be joined by piecewise geodesic segments. On the other hand, suppose $c(t)$ is a geodesic with domain $[0, L]$. As ϕ and ψ are both isometries, it follows that $\phi \circ c$ and $\psi \circ c$ are geodesics with the same initial conditions. Therefore, both curves are the same, and $\phi(c(L)) = \psi(c(L))$. Finnally, any point $x \in X$ can be joined with p by a piecewise geodesic segment and the previous argument shows that $\phi(x) = \psi(x)$. □

Theorem 3.12. *The isometry group of AdS^n is*

$$PO(n-1,2) = \left\{A \in GL(n+1,\mathbb{R}) : A^t\eta A = \eta\right\} / \{\pm I\}.$$

Proof. Since $A^t\eta A = \eta$, each $A \in O(n-1,2)$ is an isometry in the hyperboloid, and descends to another isometry in $PO(n-1,2)$. Let $\{e_1, \ldots, e_{n+1}\}$ be the canonical basis on $\mathbb{R}^{n-1,2}$. We can consider $\{e_1, \ldots, e_n\}$ as an non-degenerated orthonormal frame on e_{n+1}. Let ϕ be an isometry of AdS^n and define $f_{n+1} = \phi(e_{n+1})$, and $f'_k = d\phi \cdot e_k$, for $k = 1, \ldots, n$. We can parallel transport each f'_k in the ambient space to a tangent vector f_k at the origin and associate f'_k with $f_k \in \mathbb{R}^{n-1,2}$ in the obvious way. Let ψ be the only linear isometry sending $\{e_1, \ldots, e_{n+1}\}$ to $\{f_1, \ldots, f_{n+1}\}$. By the previous lemma, $\psi \equiv \phi$. □

Theorem 3.13. AdS^n *is homogeneous.*

Proof. We work in the hyperboloid and project to AdS^n afterwards. Let

$$(x_1, \ldots, x_{n+1}) \in \mathrm{AdS}^n.$$

Consider a transformation in $O(n-1,2)$ of the form

$$\begin{pmatrix} R_{\bar{\theta}} & 0 \\ 0 & R_\theta \end{pmatrix},$$

where $R_{\bar{\theta}}$ is a rotation in $\mathbb{R}^{n-1}$, and R_θ is a rotation in $\mathbb{R}^2$. By choosing the rotations conveniently, we can map the point to $(r', 0, \ldots, 0, r)$, where $r'^2 - r^2 = -1$. Next, choose a *Lorentz boost*,

$$\begin{pmatrix} \cosh(\theta) & \sinh(\theta) \\ \sinh(\theta) & \cosh(\theta) \end{pmatrix},$$

mapping (r', r) to $(0, 1)$. The matrix

$$\begin{pmatrix} \cosh(\theta) & 0 & \sinh(\theta) \\ 0 & I_{n-1} & 0 \\ \sinh(\theta) & 0 & \cosh(\theta) \end{pmatrix} \in O(n-1,2)$$

sends the last point to e_{n+1}. □

Corollary 3.14. *The isotropy group of* AdS^n *is isomorphic to the projective Lorentz group,* $PO(n-1,1)$.

Proof. Since AdS^n is homogeneous, we can analyze the isotropy group of e_{n+1}. A matrix in $O(n-1,2)$ keeping e_{n+1} fixed must necessarily be of the form

$$\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix},$$

where $A \in O(n-1,1)$. □

Corollary 3.15. $\mathrm{AdS}^n \cong PO(n-1,2)/PO(n-1,1)$.

Recall that a semi-Riemannian space X is *symmetric* if it is connected, and for each $p \in X$, there is an isometry $\phi : X \to X$, such that $\phi(p) = p$, and $d\phi_p = -id_p$.

Corollary 3.16. AdS^n *is symmetric.*

Proof. Consider the map $A \in O(n-1,2)$, such that $A \cdot e_k = -e_k$, for $k = 1, \ldots, n$, and $A \cdot e_{n+1} = e_{n+1}$, and the fact that AdS^n is homogeneous. □

Remark 3.17. If $A \in O(n-1,2)$, the equation $A^t \eta A = \eta$ implies $\det(A)^2 = 1$. Therefore, $PO(n-1,2) \cong SO(n-1,2)$ if n is odd.

Remark 3.18. AdS^n is homogeneous but not isotropic, since causality restricts the action of the isotropy group. However, it is a space of constant sectional curvature. By standard methods in semi-Riemannian geometry, it can be proved that its curvature is -1, *e.g.* [6], corollary 43. Given the natural embedding of the hyperboloid, double covering, AdS^n in $\mathbb{R}^{n-1,2}$, this can be proved fairly easily with *Gauss' equation.*

Lemma 3.19 (Gauss' equation in codimension one)**.** *Let $M \subset X$ be an isommetric immersion of one orientable and oriented semi-Riemannian manifold M into another, and let* $\mathrm{Rm}, \mathbf{Rm}$ *be the corresponding curvature tensors. Then,*

$$\begin{aligned}\mathbf{Rm}(X,Y,Z,W) = \mathrm{Rm}(X,Y,Z,W) &+ \langle\langle\nabla_X N, Z\rangle N, \langle\nabla_Y N, W\rangle N\rangle \\ &- \langle\langle\nabla_X N, W\rangle N, \langle\nabla_Y N, Z\rangle N\rangle,\end{aligned}$$

where N is the normal unitary *vector field of M compatible with the induced orientation.*

For a proof of Gauss' equation in the general case, see [6].

Theorem 3.20. AdS^n *is of sectional curvature -1.*

Proof. Since $\mathbb{R}^{n-1,2}$ is flat, and vectors in $T_x\mathrm{AdS}^n$ are defined by $x^t \eta p = 0$, we can identify the normal unitary to AdS^n with the identity map: $N(x) = x$. Using this and $N^2 = -1$, Gauss' equation reduces to

$$0 = \mathrm{Rm}(X,Y,Z,W) - \langle X,Z\rangle\langle Y,W\rangle + \langle X,W\rangle\langle Y,Z\rangle.$$

The result follows, since sectional curvature is calculated in non null vectors by

$$\frac{\mathrm{Rm}(X,Y,Y,X)}{||X||^2||Y||^2 - \langle X,Y\rangle^2}.$$

□

Anti de Sitter space is the Lorentzian analog to hyperbolic space, in the sense that both are spaces of constant negative curvature. Moreover, consider the following theorem:

Theorem 3.21. *Let $\widehat{\mathrm{AdS}}^n$ be the universal cover of the hyperboloid model for AdS^n. Then, $\widehat{\mathrm{AdS}}^n$ is isometric to a product $\mathbb{H}^{n-1} \times \mathbb{R}$.*

Note that, as the proof will show, the metric is *not* the semi-Riemannian product metric.

Proof. Let $x \in \mathbb{H}^n$, $t \in \mathbb{R}$. By means of the hyperboloid model, there exists $x_1, \ldots, x_n$, such that $x_n > 0$, $x \sim (x_1, \ldots, x_n)$. Map (x, t) to

$$(x_1, \ldots, x_{n-1}, x_n \cos(t), x_n \sin(t)).$$

This define a covering map from $\mathbb{H}^{n-1} \times \mathbb{R}$ to AdS^n. Since its domain is simply connected, it must be diffeomorphic to the universal cover. By pulling back the metric in AdS^n, we obtain a expression for the metric in the embedding of the cover in $\mathbb{R}^{n+1}$:

$$dx_1^2 + \cdots + dx_{n-1}^2 - dx_n^2 - x_n^2 dt^2.$$

If we denote by dx^2 the Riemannian metric in $\mathbb{H}^{n-1}$, the expression above simplifies to $dx^2 - x_n^2 dt^2$ which is manifestly Lorentzian. □

Remark 3.22. Up to universal cover, the previous theorem says that anti de Sitter space is a *warped* Lorentzian product of hyperbolic space and the real line. In fact, AdS^n is a special case of *Robertson-Walker* spaces [5].

3.1 The projective model

The construction is straightforward given the analogy with hyperbolic space. In particular, it allows us to define a boundary at infinity. Note that every line in $\mathbb{R}^{n-1,2}$ intersects AdS^n in either zero or two antipodal points.

Definition 3.23. The projective model for anti de Sitter space is given by the open patch

$$\left\{x \in \mathbb{R}^{n-1,2} : x^t \eta x < 0\right\} / scale \subset \mathbb{R}P^n,$$

with the metric induced by pulling back the metric in the hyperboloid double cover.

Definition 3.24. The *boundary at infinity* of anti de Sitter space, $\partial^\infty \mathrm{AdS}^n$, is the boundary of the closure of the projective model in $\mathbb{R}P^n$.

Note that by definition,

$$\partial^\infty \mathrm{AdS}^n = \left\{x \in \mathbb{R}P^n : x^t \eta x = 0\right\} / scale.$$

Theorem 3.25 (Asymptotic behavior of geodesics in AdS^n).

1. *A spacelike geodesic is determined by two distinct endpoints in* $\partial^\infty \mathrm{AdS}^n$.
2. *A null geodesic limits in both directions to the same point in* $\partial^\infty \mathrm{AdS}^n$.
3. *Timelike geodesics are periodic with length* 2π.

Proof. Every spacelike geodesic is of the form $Ae^{\sqrt{\lambda}\tau} + Be^{-\sqrt{\lambda}\tau}$, where A, B are null vectors in $\mathbb{R}^{n-1,2}$ with $2AB = -1$. Note that these vectors can't be collinear; therefore, A and B correspond to distinct endpoints in $\partial^\infty \mathrm{AdS}^n$, and the geodesic projects to the line spanned by both. Conversely, given two points in $\partial^\infty \mathrm{AdS}^n$, which correspond to null vectors $A, B \in \mathbb{R}^{n-1,2}$, after rescaling as appropriated, we can assume $2AB = -1$. Therefore, the points determine a spacelike geodesic, up to rescaling the speed.

The result for null geodesics is analogous, and for timelike geodesics is straightforward. □

Definition 3.26. A submanifold $M \subset X$ of a semi-Riemannian manifold X is *spacelike* if the restriction of the external metric is Riemannian. It is say to be *timelike* if the metric has at least one timelike direction.

Theorem 3.27. *Every spacelike k dimensional totally geodesic plane is isometric to* $\mathbb{H}^k$. *Every timelike non-degenerated k-dimensional totally geodesic plane is isometric to* AdS^k.

4 AdS^3

In three dimensions, there is a nice relationship between anti de Sitter space and $PSL(2,\mathbb{R})$, the space of 2×2 real matrices of determinant one.
Let $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^{2,2}$. Associate the matrix

$$X = \begin{pmatrix} x_3 + x_1 & x_2 + x_4 \\ x_2 - x_4 & x_3 - x_1 \end{pmatrix}, \tag{1}$$

then $x^t \eta x = -\det(X)$.

Theorem 4.1. *The space of* 2×2 *matrices with the product*

$$\left\langle \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \right\rangle = -\frac{1}{2}\mathrm{tr}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d' & -b' \\ -c' & a' \end{pmatrix} \right)$$

is isometric to $\mathbb{R}^{2,2}$.

Proof. If there is such isometry, then $-\det$ should be the quadratic form associated to the inner product. Recall the polarization identity:

$$\langle X, Y\rangle = \frac{||X+Y||^2 - ||X-Y||^2}{4},$$

provided the inner product exists. Writing X and Y in components, we arrive to the expression given in the theorem. That this is an inner product in the space of matrices comes from the fact that, after pulling back with (1), $\langle X, Y\rangle$ corresponds to the metric in $\mathbb{R}^{2,2}$. □

Corollary 4.2. *If* $Y \in \mathbb{R}^{2,2}$ *is represented by an invertible matrix, then*

$$\langle X, Y\rangle = -\frac{1}{2}\det(Y)\mathrm{tr}\left(XY^{-1}\right).$$

Remark 4.3. The expression in the theorem is expected, as a well-known fact from calculus says that in $GL(n,\mathbb{R})$, the derivative of det at the identity is tr. The expression we have obtained can be seen as a correction to the formula for the general case in 2×2.

Definition 4.4. $PGL(2,\mathbb{R})^{2,+}$ is the subgroup of $PGL(2,\mathbb{R})\times PGL(2,\mathbb{R})$, such that, if (A,B) is a representant of each class, then $\det(A) = \det(B) = \pm 1$.

Theorem 4.5. *Let* $A, B \in GL(2,\mathbb{R})$ *be, such that* $\det(A) = \det(B) = \pm 1$*, then the mapping*

$$(A,B)\cdot X = AXB^{-1}$$

defines an action by isometries of $PGL(2,\mathbb{R})^{2,+}$ *in* AdS^3*.*

Proof. The action is by isometries, because if $X \in \mathrm{AdS}^3$,

$$\langle (A,B)\cdot X, (A,B)\cdot Y\rangle = -\frac{1}{2}\mathrm{tr}\left(AXY^{-1}A^{-1}\right) = -\frac{1}{2}\mathrm{tr}\left(XY^{-1}\right),$$

where the last equality is due to the commutativity of the trace. □

Hence, we have a monomorphism $PGL(2,\mathbb{R})^{2,+} \to PO(2,2)$. It can be shown that each mapping preserves orientation in AdS^3, and that $PGL(2,\mathbb{R})^{2,+} \cong PSO(2,2)$. [3]

Remark 4.6. Orientation reversing isometries are given by another copy of $PGL(2,\mathbb{R})^{2,+}$, acting by $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \mapsto A\left(\begin{smallmatrix} d & -b \\ -c & a\end{smallmatrix}\right)B^{-1}$.

Recall that any 2×2 matrix with determinant 0 must be either the null matrix or have rank 1. We have proved:

Proposition 4.7.

$$\partial^\infty \mathrm{AdS}^3 = \left\{ A \in \mathbb{R}^{2\times 2} : \mathrm{rk}(A) = 1 \right\}.$$

A well-known fact from linear algebra states that, any rank one matrix can be written as vw^t, for some vectors $v, w \in \mathbb{R}^n$. Note that if v', w' is another pair of vectors, such that $vw^t = v'w'^t$, there should exist a scalar λ, such that $v = \lambda v'$, $w = \frac{1}{\lambda} w'$. Therefore, $\partial^\infty \mathrm{AdS}^3$ is diffeomorphic to $\mathbb{R}P^1 \times \mathbb{R}P^1$, which topologically is a torus.

Definition 4.8. Given $a, b \in \mathbb{R}$, define

$$\begin{pmatrix} a \\ b \end{pmatrix}^\dagger = \begin{pmatrix} -b & a \end{pmatrix}.$$

Proposition 4.9. *If $A \in GL(2, \mathbb{R})$, then $(Av)^\dagger = v^\dagger \det(A) A^{-1}$.*

Theorem 4.10. *There is a diffeomorphism $\partial^\infty \mathrm{AdS}^3 \cong \mathbb{R}P^1 \times \mathbb{R}P^1$, such that the action of the orientation preserving isometries, $Isom^+(\mathrm{AdS}^3) = PGL(2, \mathbb{R})^{2,+}$ is the product action.*

Proof. By changing signs in w, we can show that a rank one matrix can be expressed as $A = vw^\dagger$. This gives the desired isomorphism. Let $A, B \in GL(2, \mathbb{R})$ be, such that $\det(A) = \det(B) = \pm 1$, and $X = vw^\dagger$, a rank one matrix in $\partial^\infty \mathrm{AdS}^3$. Then,

$$(A, B) \cdot X = A\, vw^\dagger\, B^{-1} = Av\, (Bw)^\dagger.$$

□

In the three dimensional case, it is possible to describe the torus that corresponds to the boundary at infinity explicity. In the projective model, $\partial^\infty \mathrm{AdS}^3 \subset \mathbb{R}P^3$ is the set

$$\left\{ [x_1 : x_2 : x_3 : x_4] : x_1^2 + x_2^2 = x_3^2 + x_4^2 \right\}.$$

In the chart $x_4 = 1$, $\partial^\infty \mathrm{AdS}^3$ corresponds to the hyperboloid $x_1^2 + x_2^2 = 1 + x_3^2$, whereas AdS^3 is the region in $\mathbb{R}^3$ containing the axis x_3. As a consequence of corollary 3.7, the boundary at infinity is a ruled surface. We can give explicit parametrization of the null lines generating it, as follows:

$$\begin{aligned} x_1(s) &= -\sin(\theta)\, s + \cos(\theta), & x_3(s) &= s, \\ x_2(s) &= \cos(\theta)\, s + \sin(\theta), & x_4(s) &= 1, \end{aligned}$$

then, $x(s) = [x_1(s) : x_2(s) : x_3(s) : x_4(s)]$, $s \in \mathbb{R}$, is a curve in the boundary at infinity, and in the patch $x_4 = 1$. Observe that in projective space,

$$\lim_{s\to\pm\infty} x(s) = \left[\frac{x_1(s)}{x_3(s)} : \frac{x_2(s)}{x_3(s)} : 1 : \frac{1}{x_3(s)}\right] = [-\sin(\theta) : \cos(\theta) : 1 : 0]\,;$$

i.e., in $\mathbb{R}P^3$, $x(s)$ can be extended to a continuous closed curve for each value of the parameter $\theta \in [0, 2\pi]$, and this construction gives an explicit description of $\partial^\infty \mathrm{AdS}^3$ as a torus.

This construction can be generalized in the case of odd dimensional spheres, where a continuous, non-vanishing tangent vector field can be constructed.

Proposition 4.11. *$\partial^\infty \mathrm{AdS}^{2n+1}$ is topologically the product of a circumference and a sphere. In particular, $\partial^\infty \mathrm{AdS}^3$ is a torus.*

Proof. Let $[x' : s' : 1] \in \partial^\infty \mathrm{AdS}^{2n+1}$, where $x' \in \mathbb{R}^{2n}$, $s' \in \mathbb{R}$, and such that $||x'||^2 = s'^2 + 1$. Here, $||x'||^2$ stands for the Euclidean norm in $\mathbb{R}^{2n}$. Write x' in the cannonical basis as

$$x' = \sum_{i=1}^{n}\left(x_i' \frac{\partial}{\partial x_i} + y_i' \frac{\partial}{\partial y_i}\right).$$

Let

$$x_i = \frac{x_i' + s' y_i'}{1 + s'^2}, \qquad y_i = \frac{-s' x_i' + y_i'}{1 + s'^2}.$$

The vectors $w, v \in \mathbb{R}^{2n}$ defined by the equations

$$w = \sum_{i=1}^{n}\left(x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i}\right), \qquad v = \sum_{i=1}^{n}\left(-y_i \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial y_i}\right)$$

are both unitary and perpendicular with respect to the Euclidean inner product. Moreover, a direct calculation shows that $x' = s\,v + w$. Define $\gamma_w(s) = [s\,v + w : s : 1] \in \partial^\infty \mathrm{AdS}^{2n+1}$, and calculate the limit:

$$\lim_{s\to\pm\infty} \gamma_w(s) = \left[v + \frac{1}{s} w : 1 : \frac{1}{s}\right] = [v : 1 : 0].$$

Therefore, γ_w can be extended to a continuous, closed curve in the infinite boundary, such that the mapping $(s, w) \mapsto \gamma_w(s)$ extends to a diffeomorphism $S^1 \times S^{2n-1} \cong \partial^\infty \mathrm{AdS}^{2n+1}$. □

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Ribbon graphs and the fundamental group of surfaces *

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Abstract

In the present work, we are going to give a formal exposition of the ribbon graphs topic based on notes of Labourie [6], since is difficult to find as such in the literature. As an application, we are going to compute the fundamental group of surfaces using ribbon graphs as a combinatorial version of it.

Keywords: Ribbon graphs, fundamental group of surfaces.

1 Introduction

The ribbon graphs gain their mathematical popularity through the work of Penner [7] who introduce a cell decomposition of Riemann moduli space, which was later used in Kontsevich's proof of Witten conjecture [5]. The ribbon graphs are very useful for the study of the representation variety of surface groups $Hom(\pi_1(S), G)/G$ for a given surface S and a group G. In the present work, we are going to define the ribbon graphs, then we are going to use them to proof the classification Theorem of surface, and we are going to compute the fundamental group of a surface using the fundamental group of ribbon graphs.

2 Surfaces as 2-dimensional manifolds

Definition 2.1. A surface is a connected 2-dimensional smooth manifold.

A 2-dimensional chart for a surface S is a pair (U, ϕ), where $U \subset S$ is an open set and $\phi : U \to \phi(U) \subset \mathbb{R}^2$ is an homeomorphism on its image.

A collection of charts $\{(U_i, \phi_i)\}_{i \in I}$ is called an atlas for S if $S = \cup_{i \in I} U_i$, and we say that the atlas is smooth or C^∞ if the change of coordinates $\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$ is a smooth function for all $i, j \in I$. Given a function $f : S \to \mathbb{R}$, we say that f is smooth if $f \circ \phi_i^{-1} : \phi_i(U_i) \to \mathbb{R}$ is a smooth function from $\mathbb{R}^2$ to $\mathbb{R}$.

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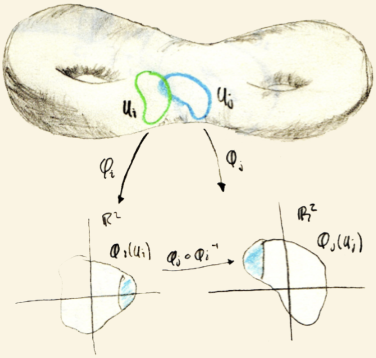

Figure 1: A surface.

Definition 2.2. Let S be a surface with atlas $\{(U_i, \phi_i)\}_{i\in I}$. The atlas is called oriented if the Jacobian

$$Jac(\phi_i, \phi_j) := det(D(\phi_i \circ \phi_j^{-1}))$$

is positive for all $i, j \in I$. Then we say that S is oriented.

2.1 Surfaces with boundary

Let $H^+ := \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$ be the closed upper half plane and $\partial H^+ := \{(x, y) \in \mathbb{R}^2 \mid y = 0\}$ its boundary.

Given a surface S, a two dimensional chart with boundary is a pair (U, ϕ), where U is an open subset of S and $\phi : U \to V \subset H^+$ is an homeomorphism into an open subset V of H^+. The subset $\partial U := \phi^{-1}(\phi(U) \cap \partial H^+) \subset U$ is the boundary of U.

Definition 2.3. Let V_1, V_2 be open subsets of H^+. A function $f : V_1 \to V_2$ is smooth if there is an open subset $\tilde{V_1}$ of $\mathbb{R}^2$ with $\tilde{V_1} \cap H^+ = V_1$, and a smooth function $\tilde{f} : \tilde{V_1} \to \mathbb{R}^2$, such that $f = \tilde{f}|_{V_1}$.

An atlas of charts with boundary $\{(U_i, \phi_i)\}_{i\in I}$ is smooth if the change of coordinates $\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$ is smooth for all $i, j \in I$ in the sense of the last definition.

Definition 2.4. A surface with boundary is a surface S with a smooth atlas of charts with boundary.

Given a surface S with boundary, we say that $x \in S$ is a boundary point if $x \in \partial S$ for any chart (U, ϕ) containing it. The set of boundary points of S is denoted by ∂S.

2.2 Gluing surfaces

We need to know how to construct new surfaces from surfaces with boundary; in order to do this, we need to glue the surfaces along their boundaries and we need to know how they look like in a neighborhood of a boundary component. For this, we have to use the following lemma:

Lemma 2.5 (Collar Lemma)**.** *Let S be a surface with boundary ∂S and $\mathcal{C} \subset \partial S$ a connected component. Then there is a neighborhood U of $\mathcal{C}$ in S and a diffeomorphism $\psi : U \to V$ into a subset $V \subset \mathbb{R}^2$ of the form $V \simeq \mathcal{C} \times [0,1]$ mapping $\mathcal{C}$ into $\mathcal{C} \times \{0\}$.*

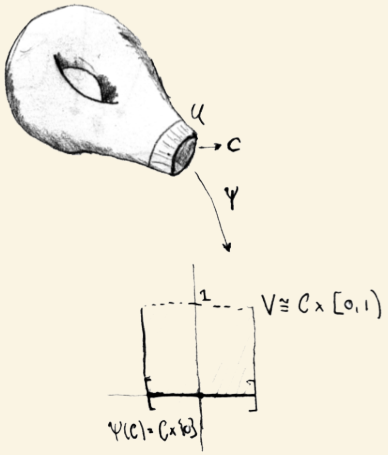

Figure 2: Collar Lemma.

Let S_1, S_2 be two surfaces with boundary. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two diffeomorphic components of ∂S_1 and ∂S_2 respectively. Then the gluing of a surface is as follows: Let $f : \mathcal{C}_1 \to \mathcal{C}_2$ be a diffeomorphism. Consider the disjoint union $S_1 \sqcup S_2$ and the following equivalence relation:

$$x \sim y \Leftrightarrow y = f(x)$$

for $x \in \mathcal{C}_1$ and $y \in \mathcal{C}_2$. Then $S_1 \cup_f S_2 = S_1 \sqcup S_2 / \sim$ and we call this quotient the gluing surface of S_1 and S_2.

The atlas of $S_1 \cup_f S_2$ is now given in the following way: First, we take a smooth atlas $\{(U_i, \phi_i)\}_{i \in I}$ for $S_1 \backslash \mathcal{C}_1$, and a smooth atlas $\{(V_j, \psi_j)\}_{j \in J}$ for $S_2 \backslash \mathcal{C}_2$. We denote by $\iota_1 : S_1 \hookrightarrow S_1 \cup_f S_2$, $\iota_2 : S_2 \hookrightarrow S_1 \cup_f S_2$ the canonical inclusions. Then $\{(\iota_1(U_i), \phi_i \circ \iota_1^{-1})\}_{i \in I} \cup \{(\iota_2(V_j), \psi_j \circ \iota_2^{-1}\}_{j \in J}$ is an atlas for the complement of the gluing curve in $S_1 \cup_f S_2$.

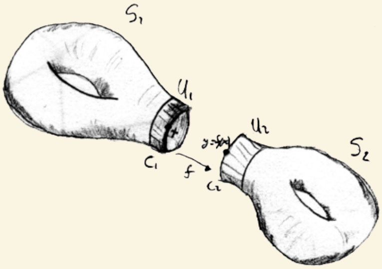

Figure 3: Gluing surfaces.

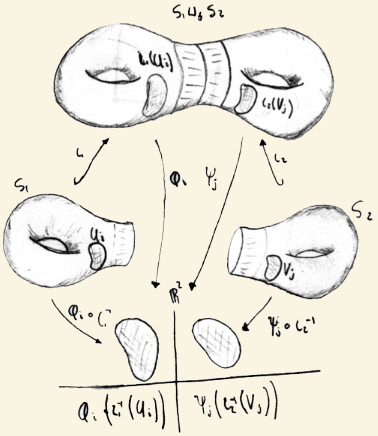

Figure 4: Atlas for the complement of the gluing curve in $S_1 \cup_f S_2$.

Now we consider a chart for the gluing curve which is compatible with the charts given above. Using the Collar Lemma we construct this charts. Let B_1, B_2 be collar neighborhoods of $\mathcal{C}_1$, $\mathcal{C}_2$ respectively, where $\mathcal{C}_i$ are connected components of ∂S_i for $i = 1, 2$, and $g_1 : B_1 \to \mathcal{C}_1 \times (-1, 0]$ a diffeomorphism, such that $g_1(\mathcal{C}_1) = \mathcal{C}_1 \times \{0\}$, and $g_2 : B_2 \to \mathcal{C}_2 \times [0, 1)$ a diffeomorphism, such that $g_2(\mathcal{C}_2) = \mathcal{C}_2 \times \{0\}$. Now consider the open subset $\mathcal{O} := B_1 \cup_f B_2$ of $S_1 \cup_f S_2$ and fix an emmbeding $\iota : \mathcal{C}_2 \times (-1, 1) \to \mathbb{R}^2$. This is possible since $\mathcal{C}_2$ is either an interval or a circle. We define coordinates for $\mathcal{O}$ by $\psi : \mathcal{O} \to \mathbb{R}^2$ as

$$\psi(x) = \begin{cases} (\iota \circ (f, id) \circ g_1)(x) & \text{if} \quad x \in B_1 \\ (\iota \circ g_2)(x) & \text{if} \quad x \in B_2, \end{cases}$$

where $f : \mathcal{C}_1 \to \mathcal{C}_2$ is a diffeomorphism.

We have proven the following proposition:

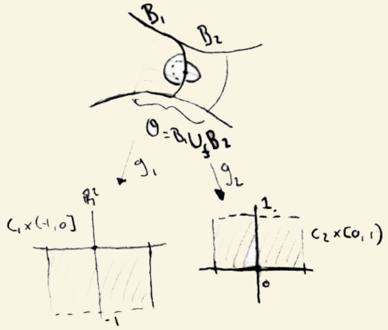

Figure 5: Atlas for the gluing curve.

Proposition 2.6. *$S_1 \cup_f S_2$ is a surface with smooth atlas given by*

$$\{(\iota_1(U_i), \phi_i \circ \iota_1^{-1})\}_{i\in I} \cup \{(\iota_2(V_j), \psi_j \circ \iota_2^{-1})\}_{j\in J} \cup \{\mathcal{O}, \psi\}.$$

Remark: Given two oriented surfaces with boundary S_1, S_2, we can give a unique orientation to $S_1 \cup_f S_2$ compatible with the orientation of S_1 and S_2 using an orientation reversing diffeomorphism $f : \mathcal{C}_1 \to \mathcal{C}_2$, where $\mathcal{C}_1$ and $\mathcal{C}_2$ are connected components of ∂S_1 and ∂S_2 respectively.

3 Surfaces as combinatorial objects

3.1 Ribbon graphs

In an informal way, a graph is a collection of points called vertex which are joined by some lines called edges as in the Figure 6. If we choose an orientation on the edges, we say that the graph is oriented or directed. The following definition gives us a formal description of these objects:

Definition 3.1. An oriented graph Γ is a triple $\Gamma = (V, E, i)$, where V is a finite set $V = \{v_1, \dots, v_n\}$ whose elements are called vertex and E is a finite set whose elements are called edges, and a map $i : E \to V \times V$ with $i(e) = (e_-, e_+)$, where e_- is the origin of the edge e and e_+ is the end of the edge e.

We say that an edge and a vertex are incident if the vertex is on the image of the edge under the map i. The quantity $a_{jk} = |i^{-1}(v_j, v_k)|$ gives us the number of edges that connect two vertex v_j and v_k.

The degree or valence of a vertex v_j is the number given by

$$deg(v_j) = \sum_{k\neq j} a_{jk} + 2a_{jj},$$

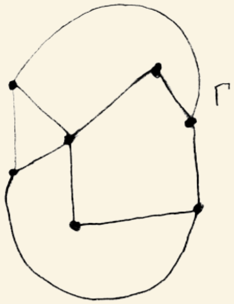

Figure 6: Example of a graph Γ.

which is the number of edges incident to v_j. A loop, this is an edge with just one incident vertex, contributes twice to the degree.

Definition 3.2. The edge refinement of an oriented graph $\Gamma = (V, E, i)$ is the graph $\Gamma_E = (V \sqcup V_E, E \sqcup E, i_E)$ with a point added at each edge as a degree 2 vertex, where V_E denotes the set of this vertices. The set of vertices of Γ_E is $V \sqcup V_E$ and the set of edges is $E \sqcup E$. The incidence relation is described by the map $i_E : E \sqcup E \to V \times V_E$ because each edge of Γ_E connects exactly one vertex of V to a vertex of V_E, and an edge of Γ_E is called a half-edge.

For each vertex $v \in V$ of Γ_E, the set $i_E^{-1}(\{v\} \times V_E)$ consists of half-edges incident to v and we have $deg(v) = |i_E^{-1}(\{v\} \times V_E)|$.

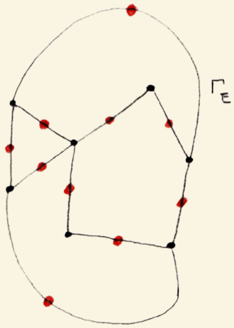

Figure 7: Edge refinement Γ_E of the graph Γ.

Remark: Let $e \in E$ be an edge of Γ, then $i(e) = (e_-, e_+)$. We denote by e_0 the vertex added on the edge e in the refinement of Γ, *i.e,* $e_0 \in V_E$ and we denote by e^- and e^+ in $E \sqcup E$ the edges, such that $i_E(e^-) = (e_-, e_0)$ and $i_E(e^+) = (e_0, e_+)$.

Let $\Gamma = (V, E, i)$ be an oriented graph and let $I : E \to E$ be an involution map with $I(e) = \bar{e}$, where $\overline{e_-} = e_+$ and $\overline{e_+} = e_-$. We call the pair $(e, \bar{e}) \in E \times E$ a geometric edge of the graph (Γ, I).

The geometric realization $|\Gamma|_I$ of the graph (Γ, I) is a topological space $|\Gamma|_I = E \times [0,1]/\sim$, where $\sim$ is the equivalence relation generated by the relations:

- $(e, t) \sim (\bar{e}, 1-t)$
- If $e, f \in E$ with $e_- = f_-$, then $(e, 0) \sim (f, 0)$
- If $e, f \in E$ with $e_+ = f_+$, then $(e, 1) \sim (f, 1)$

Remark: The geometric realization of a graph $\Gamma = (V, E, i)$ not always can be drawn on a plane $\mathbb{R}^2$ without intersections; however, we can draw the geometric realization of a graph on $\mathbb{R}^3$. To do this, let $p : \mathbb{R} \to \mathbb{R}^3$ be the function $p(t) = (t, t^2, t^3)$, and C be the curve $C = \{p(t) : t \in \mathbb{R}\}$. Now we only need to take any vertex $v_i \in V$ into the curve, and to see that the edges do not intersect, we need only show that given four vertex on C they are not coplanar. Now for any four points v_1, v_2, v_3, v_4 in $\mathbb{R}$, the volume of the tetrahedron T formed by $p(v_i) \in C$ is proportional to a Vandermonde determinant:

$$\begin{aligned} 6Vol(T) &= det((p(v_2) - p(v_1)) \cdot [(p(v_3) - p(v_1)) \times (p(v_4) - p(v_1))]) \\ &= det \begin{bmatrix} 1 & v_2 & v_1^2 & v_1^3 \\ 1 & v_2 & v_2^2 & v_2^3 \\ 1 & v_3 & v_3^2 & v_3^3 \\ 1 & v_4 & v_4^2 & v_4^3 \end{bmatrix} \neq 0, \end{aligned}$$

this implies that any four points on C are not coplanar. As a result, the edges of the tetrahedron T intersect only the appropriate vertex. Now we take arbitrary n distinct points p_i in C. The argument above shows that if we form the graph Γ from this n points, the edges intersect only in the appropriate vertex and this gives us an embedding of the given graph Γ into $\mathbb{R}^3$.

Now using this fact, we can project the graph into the plane in a such way that the edges cross over or under as in Figure 8.

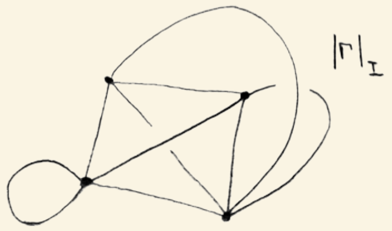

Figure 8: Geometric realization of a graph with under and over crossings.

In the same way, we can make the geometric realization of the refinament of Γ.

Now, let's we define morphisms between graphs.

Definition 3.3. A traditional graph isomorphism $\phi = (\alpha, \beta)$ between two graphs $\Gamma_1 = (V_1, E_1, i_1)$ and $\Gamma_2 = (V_2, E_2, i_2)$ is a pair of bijective maps:

$$\alpha : V_1 \to V_2, \;\; \beta : E_1 \to E_2$$

that preserves the incidence relation, *i.e.,* the following diagram commutes:

$$\begin{array}{ccc} E_1 & \xrightarrow{i_1} & V_1 \times V_1 \\ \beta \downarrow & & \downarrow \alpha\times\alpha \\ E_2 & \xrightarrow[i_2]{} & V_2 \times V_2 \end{array}.$$

Theorem 3.4. *Let* (Γ_1, I) *and* (Γ_2, I) *be isomorphic graphs, then* $|\Gamma_1|_I$ *and* $|\Gamma_2|_I$ *are homeomorphics.*

Proof. Let $\phi = (\alpha, \beta) : \Gamma_1 \to \Gamma_2$ be an isomorphism of graphs with $\beta(e) = f$ or $\beta(e) = \bar{f}$. Now, consider E_j with $j = 1, 2$ with the discrete topology; since ϕ is an isomorphism, we have that $\beta : E_1 \to E_2$ is an homeomorphism and we define the function

$$\beta \times id : E_1 \times [0,1] \to E_2 \times [0,1],$$

and we have the quotient maps $q_j : E_j \times [0,1] \to |\Gamma_j|_I$ with $j = 1, 2$. To see that this maps induces an homeomorphism on the geometric realization we just need to show that the following diagram commutes and the functions are continuous:

$$\begin{array}{ccc} E_1 \times [0,1] & \xrightarrow{\beta\times id} & E_2 \times [0,1] \\ q_1 \downarrow & & \downarrow q_2 \\ |\Gamma_1|_I & \dashrightarrow & |\Gamma_2|_I \end{array}$$

To do this, we define the function $|\phi| : |\Gamma_1|_I \to |\Gamma_2|_I$ which maps $[(e,t)] \mapsto [(\beta(e), t)]$, where $[\ ]$ denotes the equivalence class. Let's see that $|\phi|$ is well defined: If we have $[(e,t)]$, then $(e,t) \sim (\bar{e}, 1-t)$ takes $(\bar{e}, 1-t)$. Now since $\beta(e) = f$ or $\beta(e) = \bar{f}$, then $\beta(\bar{e}) = \bar{f}$ or $\beta(\bar{e}) = f$. Suppose that $\beta(e) = f$, the other case is similar, then $\beta(\bar{e}) = \bar{f}$, but $(\bar{f}, 1-t) \sim (f, t) = (\beta(e), t)$; therefore, $[(\beta(\bar{e}), 1-t)] = [(\beta(e), t)]$. Now since the diagram in the definition 2.3 commutes, we have that $[(e, 0)] \mapsto [(\beta(e), 0)]$ and $[(e, 1)] \mapsto [(\beta(e), 1)]$; therefore, the map is well defined and the diagram commutes. Let's see that the function

$|\phi|$ is continuous. Let $U \subset |\Gamma_2|_I$ be an open subset, since q_2 is a quotient map, therefore, continuous, we have that $q_2^{-1}(U)$ is open in $E_2 \times [0,1]$, and $\beta \times I$ is continuous; then $(\beta \times id)^{-1}(q_2^{-1}(U))$ is an open subset of $E_1 \times [0,1]$ and we have that q_1 is a quotient map, therefore, an open map; then $q_1((\beta \times [0,1])^{-1}(q_2^{-1}(U)))$ is open in $|\Gamma_1|_I$; therefore $|\phi|$ is continuous and since $\beta \times id$ is an homeomorphism, then $|\phi|$ is an homeomorphism. □

Now, let's consider graphs with more structure. To do this, we need the notion of cyclic ordering on a finite set S.

Definition 3.5. A cyclic ordering in a finite set S is a bijection $\sigma : S \to S$, such that for all $x \in S$ the orbit $\{\sigma^n(x)\}_{n \in \mathbb{Z}} = S$. Given $x \in S$, we will call $\sigma(x)$ the successor of x and $\sigma^{-1}(x)$ the predecessor of x.

Definition 3.6 (Ribbon graph)**.** Let (Γ, I) be a graph. For $v \in V$, the star of v

$$E_v = \{e \in E \ : \ v = e_-\}$$

is the set of edges starting from v. A ribbon graph is the graph (Γ, I), together a cyclic ordering on the star of every vertex.

Figure 9: Ribbon graph.

Remark:

1. We can see the star of a vertex $v \in V$ as the set of semi-edges starting on v when we consider the refinement of the graph and we denote this set as E_v^*.

2. We can consider isomorphisms of ribbon graphs. We just need to ask that preserve the cyclic ordering on each star, this is, that the following diagram commutes:

$$\begin{array}{ccc} E_v^1 & \xrightarrow{\beta} & E_w^2 \\ \downarrow{\scriptstyle \sigma_v^1} & & \downarrow{\scriptstyle \sigma_w^2} \\ E_v^1 & \xrightarrow[\beta]{} & E_w^2 \end{array},$$

where $\beta : E \to E$ is a bijection from the edges of the graphs

$$\phi = (\alpha, \beta) : \Gamma_1 = (V_1, E_1, i_1, I) \to \Gamma_2 = (V_2, E_2, i_2, I)$$

and σ_v^1 is the cyclic ordering on E_v^1, and σ_w^2 is the cyclic ordering on E_w^2. This isomorphism induces an homeomorphism on the geometric realization of the ribbon graph preserving the cyclic ordering on it.

For all $v \in V$, we can consider an embedding on $\mathbb{R}^2$ of the geometric realization of E_v^*; the orientation of $\mathbb{R}^2$ induces a cyclic ordering on each star in the following way: let's consider a circle with center on v and radius one (we can suppose this because we can consider the length of each edge on $|E_v^*|_I$ as 1), since the circle gets an orientation from $\mathbb{R}^2$ and we can define the cyclic ordering from this orientation (see Figure 9).

Lets consider surfaces from the ribbon graphs. In order to do this we need first to embed the geometric realization of the graph in a open oriented surface (this is not always possible to do in the plane).

Lemma 3.7. *Every ribbon graph can be embedded in an open oriented surface such that its cyclic ordering are induced from the orientation of the surface.*

Proof. We construct the surface in the following way: Let $v \in V$ be a vertex of (Γ,I) and $|E_v^*|_I$ the geometric realization of the star at v in the refinement of (Γ,I); consider an embedding of $|E_v^*|_I$ in $\mathbb{R}^2$ and take a disc $D(v)$ with center on v and radius 1 (we can consider this length for each edge in $|E_v^*|_I$). Now we take a tubular neighborhood $U_v \subset D(v)$ of $|E_v^*|_I$ with many boundary components as elements in E_v^* labelled in the following way by the elements of E_v. Start with an arbitrary edge $e \in |E_v^*|_I$, then the following boundary component is labelled by $\sigma_v(e)$, where σ_v is the cyclic ordering of $|E_v^*|_I$ and so on.

Since we do this for all vertex $v \in V$, using the Gluing Lemma, we now glue each star with the other ones in the following way: For $e \in |E_v^*|_I$ and $e' \in |E_v'^*|_I$, we glue their boundary components if we have that $i_E(e) = (v, e_0)$ and $i_E(e') = (e_0, v')$, this is, that $e = e^+$ and $e' = e^-$ in the refinement of (Γ, I) (see Figure 11), and we make this gluing, such that the orientations are reversed, then we get an oriented open surface. Since in each vertex the ordering of its corresponding star is preserved for the process of gluing, then the orientation of the surface is compatible with σ_v. □

The surface constructed in the proof of Lemma 3.7 is called the associated ribbon surface of the graph.

We can associate different ribbon graphs to a one graph, all depends on the cyclic ordering in the edges (see Figure 12) and we get different associated ribbon surfaces (see Figure 13).

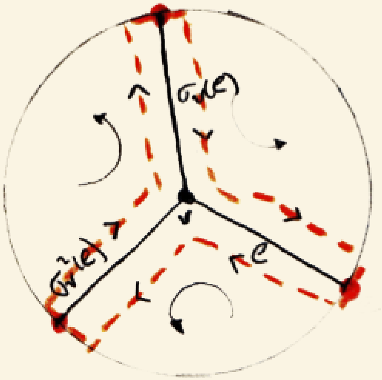

Figure 10: Tubular neighborhood of the star and its orientation.

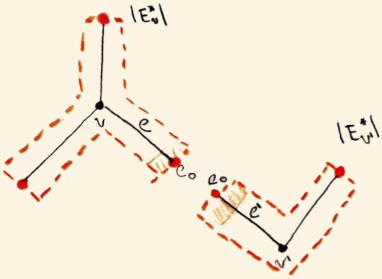

Figure 11: Gluing of the stars.

In order to embed ribbon graphs into closed surfaces, we need to close the holes in the associated ribbon surface. To do this, we define the faces of the associated ribbon surface.

Definition 3.8. Let $(\Gamma = (V, E, i), I)$ be a ribbon graph. A face is an n-tuple $(e_1, \ldots, e_n)$ of edges, such that $e_p^+ = e_{p+1 \mod n}^-$ and $\sigma_{e_p^+}(\bar{e_p}) = e_{p+1 \mod n}$ for all $1 \leq p \leq n$, where $\sigma_{e_p^+}$ is the cyclic ordering on the star of e_p^+.

The boundaries of this faces will be the boundaries of the disc we will be attaching at our associated ribbon surface.

Definition 3.9. A graph Γ embedded in a surface S is filling if each connected component of $S \backslash |\Gamma|_I$ is diffeomorphic to a disc.

Now we have:

Proposition 3.10. *Every ribbon graph (Γ, I) has a filling embedding into a compact oriented surface S. The connected components of $S \backslash |\Gamma|_I$ are in bijection with the faces of the associated ribbon surface of Γ.*

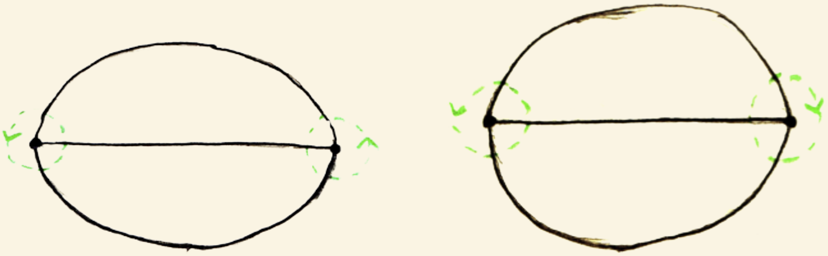

Figure 12: Ribbon graphs Γ_1 and Γ_2 from the same graph Γ.

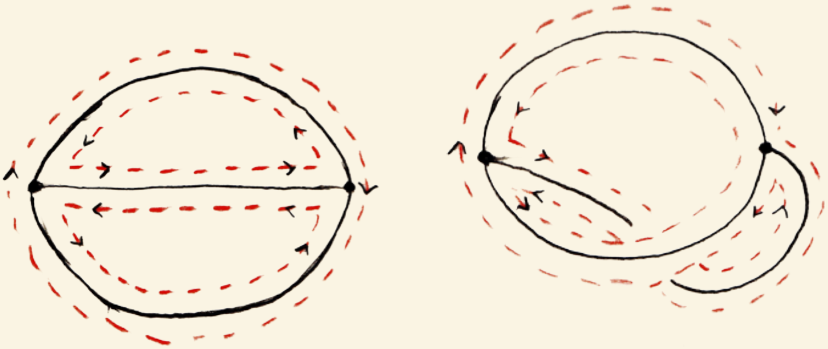

Figure 13: Associated ribbon surfaces of Γ_1 and Γ_2.

Proof. Since Γ is a ribbon graph, we have that the boundary components of the associated ribbon surface of Γ define closed curves homeomorphic to a circles. We glue a disc for each of these curves. Therefore, we thus obtain a closed surface and is followed immediately that the connected components of $S\backslash|\Gamma|_I$ are in bijection with the faces of the associated ribbon surface. □

We will see that the surface obtained from the proposition 3.10 is unique in a very strong sense. To see this, we will need the following basic fact from point-set topology:

Lemma 3.11 (Clutching Lemma)**.** *Let $X = U \cup V$ be a decomposition of a topological space X in two closed sets U and V. If $f_1 : U \to Y$ and $f_2 : V \to Y$ are continuous maps from U and V into some topological space Y, such that $f_1|_{U\cap V} = f_2|_{U\cap V}$, then the induced map $f : X \to Y$ is continuous.*

Using this, we can show the following result:

Proposition 3.12. *Let $\Gamma_1 \subset S_1$ and $\Gamma_2 \subset S_2$ be filling ribbon graphs of compact oriented surfaces, and let $\phi : \Gamma_1 \to \Gamma_2$ be an isomorphism of ribbon graphs. Then ϕ induces an homeomorphism on the geometric realization $|\phi| : |\Gamma_1|_I \to |\Gamma_2|_I$ and this extends to an homeomorphism between S_1 and S_2.*

Proof. Since ϕ is an isomorphism of ribbon graphs, then by Theorem 3.4 this extends to an homeomorphism of the geometric realization $|\phi| : |\Gamma_1|_I \to |\Gamma_2|_I$. Let S_{Γ_1} and S_{Γ_2} be the associated ribbon surfaces of Γ_1 and Γ_2 respectively, then by the Clutching Lemma this homeomorphism extends to an homeomorphism of the closure of the associated ribbon surfaces.

Since Γ_1 and Γ_2 are filling, we have that

$$S_1\backslash|\Gamma_1|_I = \sqcup_{f\in F} D_f,\ S_2\backslash|\Gamma_2|_I = \sqcup_{g\in G} D'_g,$$

where D_f and D'_g are discs. Then, we have that

$$S_1 = \overline{S_{\Gamma_1}} \cup (\sqcup_{f\in F} \bar{d}_f),\ S_2 = \overline{S_{\Gamma_2}} \cup (\sqcup_{g\in G} \bar{d}'_g),$$

where d_f and d'_g are slightly smaller discs than D_f and D'_g, and

$$\overline{S_{\Gamma_1}} \cap (\sqcup_{f\in F} \bar{d}_f) = \sqcup_{f\in F} \partial \bar{d}_f,\ \overline{S_{\Gamma_2}} \cap (\sqcup_{g\in G} \bar{d}'_g) = \sqcup_{g\in G} \partial \bar{d}'_g$$

are the unions of discs. By the Clutching Lemma is suffices to construct for each $f \in F$ an homeomorphism from d_f to d'_g, which agrees on the boundary $\partial \bar{d}_f$ with the extension $|\phi|$ to $\overline{S_{\Gamma_1}}$. But if $\psi : \mathbb{S}^1 \to \mathbb{S}^1$ is any homeomorphism of circles, then there is an obvious way to extend it to the corresponding disc. In fact, each x in the disc may be written in polar coordinates as $x = re^{i\theta}$ for some $r \in [0, 1]$ and some $e^{i\theta} \in \mathbb{S}^1$. Then, we can simply define $\psi(re^{i\theta}) = r|\phi|(e^{i\theta})$ to obtain the desired homeomorphism. □

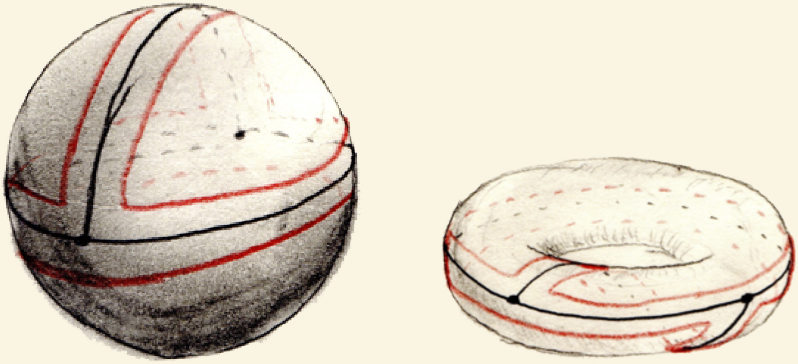

Figure 14: Closed surfaces from the ribbon graphs Γ_1 and Γ_2.

Combining propositions 3.10 and 3.12, we have:

Corollary 3.13. *For any ribbon graph Γ there exists a unique compact oriented surface S_Γ (up to homeomorphism), such that Γ can be embedded as a filling ribbon graph into S_Γ.*

Corollary 3.13 will enable us to classify surfaces up to homeomorphism and allows us to construct surfaces from ribbon graphs. The following proposition is a converse of this corollary:

Proposition 3.14. *Every compact oriented surface admits a filling ribbon graph.*

The proof of this proposition is similar to find a triangulation of the surface. Then by a Theorem of Cairns and Whitehead, we have that every smooth manifold admits a triangulation; since every surface is a two dimensional smooth manifold we are done (See [2] and [8]).

4 Classification of surfaces I: Existence

By Corollary 3.13 a convenient description of the compact oriented surfaces is given by their underlaying filling ribbon graph. We consider a family Γ_g, $g \geq 1$ of filling ribbon graphs given in the following way: Take g-copies of the graph Γ_1 shown in Figure 15, we call this graph a petal.

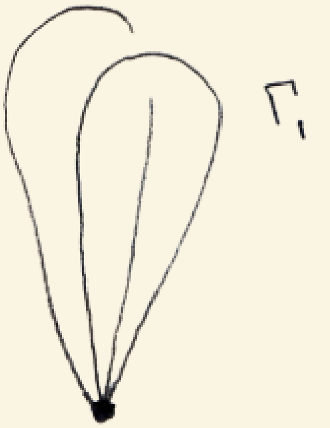

Figure 15: A petal Γ_1.

Then we glue g-copies of the petal by they vertex to get a graph Γ_g with g petals, as is shown in the following figure:

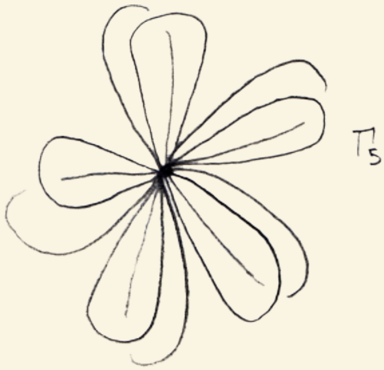

Figure 16: 5-petals ribbon graph Γ_5.

Using Definition 3.8, we can see that Γ_1 has one face and with some mental gymnastics, we can see that the associated oriented closed surface $S_1 := S_{\Gamma_1}$ is a torus.

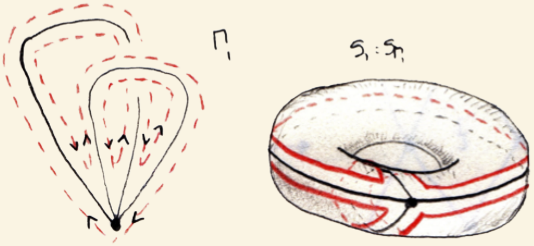

Figure 17: Surface S_1.

Similarly, each copy of Γ_1 in Γ_g is a torus with one puncture and we glue two consecutive torus by their punctures. Thus, we get a surface $S_g := S_{\Gamma_g}$, which is a handlebody with g handles.

There is another useful description of S_g as follows: Let $D_g = S_g \backslash |\Gamma_g|$ be the disc in S_g, which corresponds to the only face of Γ_g. Then, S_g is obtained by gluing the boundary of the disc D_g in the following way: Let a_i, b_i be two edges of the i-th copy of Γ_1 in Γ_g. Since each oriented edge of Γ_g, this is, $a_i, \overline{a_i}, b_i, \overline{b_i}$, occurs only once in the boundary of D_g, then we can describe the boundary of D_g by the series of edges given by

$$a_1, b_1, \overline{a_1}, \overline{b_1}, \ldots, a_g, b_g, \overline{a_g}, \overline{b_g}.$$

We can see this for the case $g = 2$ as is shown in Figure 18:

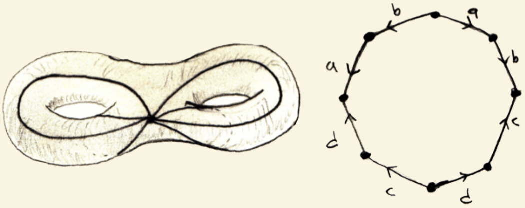

Figure 18: The surface S_2 and its gluing polygon.

It is convenient to define $S_0 := \mathbb{S}^2$, the two-sphere. Later we see that given any filling ribbon graph, we can deform it to any of the Γ_g graphs. Now we can state the first part of the Classification Theorem.

Theorem 4.1. *Every oriented compact surface S is homeomorphic to one of the surfaces S_g for $g \geq 0$.*

Proof. By Proposition 3.14, we can choose a filling ribbon graph Γ for the surface S. If Γ doesn't have edges, then S must be the surface S_0. Thus, we may assume that Γ has at least one edge. Now, we will deform the graph Γ to obtain one of the graphs Γ_g without changing the filling property in the process. In this way the theorem follows from Corollary 3.13.

First we deform the graph Γ, so we see that the surface is obtained from gluing the boundary of a polygon, in the following way:

1. Eliminating faces: Let's assume that Γ has more than one face. Then, there is a geometric edge $(e, \bar{e})$, such that e and $\bar{e}$ are in different faces. Let Γ' be the graph obtained by eliminating from Γ the edges e and $\bar{e}$, this is, $\Gamma' = \Gamma \backslash \{e, \bar{e}\}$, then Γ' is still filling and has one face less than Γ, as is shown in Figure 19:

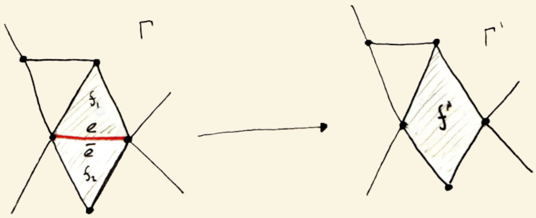

Figure 19: Eliminating edges.

   If we iterate this process, we get a filling ribbon graph with only one face.

2. Eliminating vertices: Let $\Gamma = (V, E, I)$ be a filling ribbon graph and $\phi : |\Gamma|_I \hookrightarrow S$ be an embedding on the surface S. If Γ has more than one vertex, lets say e_- and e_+, then there is an edge e joining them. Let $\Gamma' = (V', E', I)$ be a new filling ribbon graph and $\phi' : |\Gamma'|_I \hookrightarrow S$ an embedding in S, where:

   - The new set of vertices is obtained by crushing the vertex e_- and e_+ in a single vertex e_c, thus $V' = (V \backslash \{e_-, e_+\}) \cup \{e_c\}$.
   - The new set of edges is given by $E' = E \backslash \{e, \bar{e}\}$.
   - The map $\phi' : |\Gamma'| \hookrightarrow S$ is defined by sending the new vertex into a point on the geometric image of $\phi(e)$, and is extended to the edges which previously started from $e_{\mp}$.

   Again this process does not change the filling property and reduce the number of vertices by one without increasing the number of faces, as in the following figure.

   Iterating this process, we obtain a graph with only one vertex.

Then by (1) and (2), we can assume that the graph Γ has only one vertex and one face. Therefore, we get that S is obtained by gluing the sides of a polygon labelled

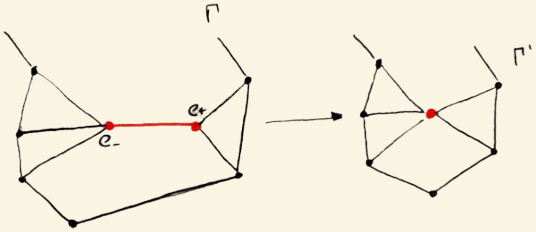

Figure 20: Eliminating vertices.

by the edges of the graph and by definition of face, every oriented edge appears once, then the gluing is given by identifying e and $\bar{e}$ with reversed orientation.

If there are no edges left, then S is the surface S_0 and we are done. Thus, assume that Γ has at least one edge. Let $(a, \bar{a})$ and $(b, \bar{b})$ geometric edges from Γ. We will call the pair $((a, \bar{a}), (b, \bar{b}))$ linked if their relative position is as in the following figure:

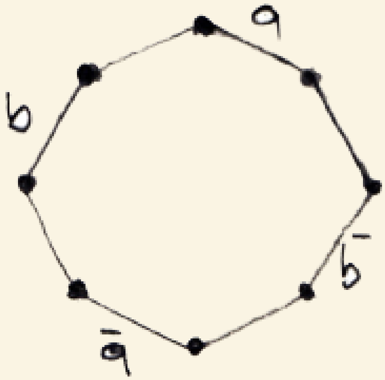

Figure 21: Linked geometric edges.

Claim: Any geometric edge of Γ is linked to at least other geometric edge. *Proof*: Assume that $(a, \bar{a})$ is not linked to any other edge, then this edge would produce an additional face since Γ is a ribbon graph, but this contradicts the assumption that there is only one face.

The following claim let us rearrenge the labelling of the sides of the polygon in such way that we obtain a graph Γ_g:

Claim: Given a linked pair $(a, \bar{a})$, $(b, \bar{b})$ of geometric edges, there is a way of rearrenging the labelling of the polygon without changing the resulting quotient space, such that

- $a, b, \bar{a}, \bar{b}$ appears as a subsequence of the sides of the polygon.

- no subsequence of type $c, d, \bar{c}, \bar{d}$ is destroyed during this process.

Proof: First we add an edge to obtain two faces as is shown in Figure 22.

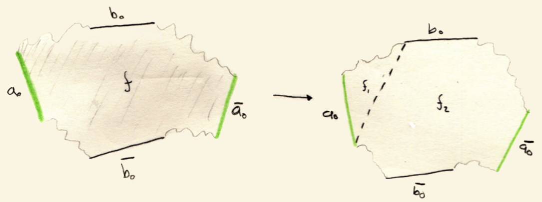

Figure 22: Adding an edge.

Then we erase in the graph the green edges, which has the effect on the polygon to glue together these two green lines in the red one (see Figure 23):

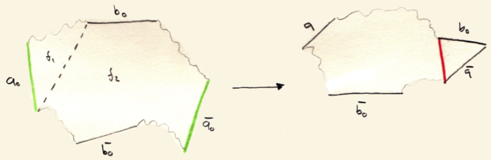

Figure 23: Eliminating the green edges.

Then we repeat this procedure two more times as is depicted in Figure 24. In the final picture, we have created an additional subsequence of the form $a, b, \bar{a}, \bar{b}$, which proves the claim.

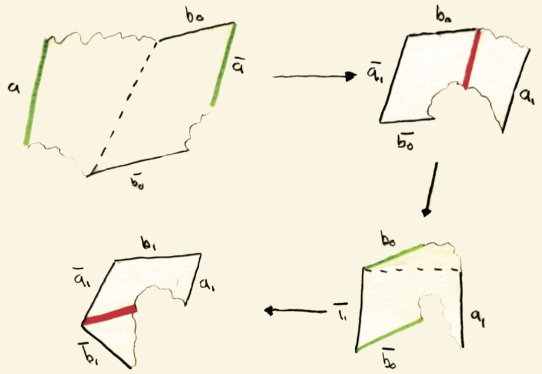

Figure 24: Eliminating the green edges.

Then the resulting surface (which is homeomorphic to S) is thus brought into a new position, such that all the edges of its polygon are of the form

$$a_1, b_1, \overline{a_1}, \overline{b_1}, \ldots, a_g, b_g, \overline{a_g}, \overline{b_g}$$

and we conclude the proof of the theorem. □

5 The fundamental group of a surface

In this section, for the proofs of the results, we refer the lector to the books [4] and [1].

5.1 The fundamental group of a topological space

At this moment, we have proved half of the Classification Theorem; in order to prove the other half, we need to know how to distinguish two surfaces S_g and $S_{g'}$ when $g \neq g'$. In order to show this, we need an invariant that distinguishes the surfaces S_g and $S_{g'}$ from each other.

This invariant is the fundamental group and we briefly recall its definition:

Definition 5.1. Let X, Y be topological spaces:

- A parametrised loop in X based at $x_0 \in X$ is a continuous map $\gamma : [0,1] \to X$ with $\gamma(0) = x_0 = \gamma(1)$. We denote by $\Omega(X, x_0)$, the set of loops in X based at x_0.

- The composition of two based loops $\gamma_0, \gamma_1 \in \Omega(X, x_0)$ is defined as:

$$(\gamma_0 * \gamma_1)(t) = \begin{cases} \gamma_0(2t), & 0 \leq t \leq \frac{1}{2} \\ \gamma_1(2t-1), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

- Let $f_0, f_1 : Y \to X$ be continuous maps which agree on a subset $A \subset Y$. Then, f_0 and f_1 are called homotopic relative to A, denoted by $f_0 \simeq f_1(relA)$, if there exists a map $H : [0,1] \times Y \to X$ with

$$H(0, y) = f_0(y)$$

$$H(1, y) = f_1(y)$$

$$\forall a \in A,\ H(t, a) = f_0(a) = f_1(a)$$

the map H is called homotopy relative to A. A space X is called contractible if the identity map $id : X \to X$ is homotopic to a constant map $x \mapsto x_0$ for some $x_0 \in X$.

- Two based loops $\gamma_0, \gamma_1 \in \Omega(X, x_0)$ are called homotopic if there is a homotopy relative to $\{0, 1\}$. The set $\pi_1(X, x_0) = \Omega(X, x_0)/\sim$, where $\sim$ is the equivalence relation given by $\gamma_0 \sim \gamma_1 \Leftrightarrow \gamma_0 \simeq \gamma_1 (rel\{0, 1\})$.

Theorem 5.2. *The set $\pi_1(X, x_0)$ is a group with the operation given by $* : \pi_1(X, x_0) \times \pi_1(X, x_0) \to \pi_1(X, x_0)$, $[\gamma_0] * [\gamma_1] = [\gamma_0 * \gamma_1]$, where $[\ ]$ denotes a homotopy class and $*$ is the composition of loops.*

Definition 5.3. The group $(\pi_1(X, x_0), *)$ is called the fundamental group of the space X based on x_0.

Let $x_0, x_1 \in X$ and let $\gamma : [0, 1] \to X$ with $\gamma(0) = x_0$, $\gamma(1) = x_1$. Then, the map

$$\pi_1(X, x_0) \to \pi_1(X, x_1), [\alpha] \mapsto [\gamma * \alpha * \bar{\gamma}]$$

is an isomorphism, where the composition of paths is defined as composition of loops above and $\overline{\gamma}(t) := \gamma(1 - t)$. The isomorphism type of the fundamental group of an arcwise connected space X not depends on the base point.

Definition 5.4. An arcwise connected space X is called simply-connected if $\pi_1(X, x_0)$ is trivial for some (and hence any) $x_0 \in X$.

Examples:

1. $\pi_1(x_0) = \{1\}$, where x_0 is a point.
2. $\pi_1(\mathbb{S}^1) = \mathbb{Z}$.
3. $\pi_1(\mathbb{S}^1 \times \mathbb{R}) = \mathbb{R}$.
4. $\pi_1(\mathbb{S}^n) = \{1\}$ for $n \geq 2$.
5. $\pi_1(\infty) = F_2$, the free group on two generators.

5.2 Coverings and the fundamental group

Computing the fundamental group using only the definition is in many cases impossible. One common way to compute the fundamental group is by looking the space as a quotient of a simply-connected space. To do this, we need the following notions:

Definition 5.5 (Group action)**.** Let G be a group and S a nonempty set. Then, G is said to act on S if there is function from $G \times S$ to S, usually denoted $(g, s) \mapsto gs$, such that for the identity $e \in G$, $es = s$ for all $s \in S$, and for all $g, h \in G$ and $s \in S$, $(gh)s = g(hs)$.

Remark: The previous definition is for left actions, but we can define right actions as follows: $S \times G \to S$ with $(s, g) \mapsto sg$ and with the same properties.

Definition 5.6. Suppose that G is a group which acts on a set S. If $s \in S$, let $G(s) = \{gs | g \in G\}$. The set $G(s)$ is called the orbit of s. The stabilizer of s is the subset $G_s = \{g \in G | gs = s\}$.

Definition 5.7. Let Γ be a discrete group which acts on a space M. Then the action is called free if it has no fixed points; in other words, the stabilizer Γ_x is trivial for all $x \in M$. The action is properly discontinuous if for any compact set $K \subset M$ the set

$$\Gamma_K = \{\gamma \in \Gamma | \gamma K \cap K \neq \emptyset\}$$

is finite.

The reason of why we need these tools is the following:

Theorem 5.8. *Let Γ act on a space X proper discontinuously. Then X is Housdorff if and only if X/Γ is Housdorff.*

Now we introduce some basic notions of covering spaces.

Definition 5.9. A covering space of a space X is a space $\tilde{X}$ together with a map $p : \tilde{X} \to X$, such there is an open cover $\{U_\alpha\}_{\alpha \in I}$ of X, such that for each α, $p^{-1}(U_\alpha)$ is a disjoint union of open sets in $\tilde{X}$, each of which is mapped by p homeomorphically onto U_α.

Definition 5.10. Given a covering $p : \tilde{X} \to X$, a lifting of a map $f : Y \to X$ is a map $\tilde{f} : Y \to \tilde{X}$, such that $f = p \circ \tilde{f}$.

Proposition 5.11 (Homotopy lifting property)**.** *Given a covering space $p : \tilde{X} \to X$, a homotopy $f_t : Y \times [0, 1] \to X$, and a lifting $\tilde{f}_0 : Y \to \tilde{X}$ of f_0, there is a unique homotopy $\tilde{f}_t : Y \times [0, 1] \to \tilde{X}$ that lifts f_t.*

Proposition 5.12. *The induced map $p_* : \pi_1(\tilde{X}, \tilde{x_0}) \to \pi_1(X, x_0)$ is injective. The image subgroup $p_*(\pi_1(\tilde{X}, \tilde{x_0}))$ consists of homotopy classes of loops in X based at x_0 that lift to loops in $\tilde{X}$ based at $\tilde{x_0}$.*

Definition 5.13. A space X is semilocally simply connected if each point $x \in X$ has a neighborhood U, such that $\pi_1(U, x) \subset \pi_1(X, x)$ is trivial.

Theorem 5.14. *If a space X is path connected and locally path connected, then X has a simply connected covering space if and only if X is semilocally simple-connected.*

Theorem 5.15. *If $\tilde{X}_1 \to X$ is a covering space and $\tilde{X} \to X$ is a simple-connected covering space, then $\tilde{X}$ is a covering space of $\tilde{X}_1$. Thus, there is a partial ordering of covering spaces.*

The simply-connected covering space $\tilde{X}$ of X is called the universal covering of X. We will be only interested on Universal covers.

We introduce some basic facts about deck transformations.

Definition 5.16. An (self) isomorphism of covering spaces $\tilde{X} \to \tilde{X}$ is called a deck transformation. These forms a group $G(\tilde{X})$.

Definition 5.17. A covering space $p : \tilde{X} \to X$ is normal if for each $x \in X$ and each pair of lifts $\tilde{x}, \tilde{x}' \in p^{-1}(x)$, there is a deck transformation taking $\tilde{x}$ to $\tilde{x}'$.

Proposition 5.18. *Let $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ be a path-connected covering space of a path-connected, locally path-connected space X, and let*

$$H = p_*(\pi_1(\tilde{X}, \tilde{x_0})) \leq \pi_1(X, x_0).$$

Then:

1. *The group of deck transformations $G(\tilde{X})$ is isomorphic to $N(H)/H$, where $N(H)$ is the normalizer subgroup.*

2. *The covering space is normal if and only if H is a normal subgroup of $\pi_1(X, x_0)$.*

Corollary 5.19. *If $\tilde{X}$ is a normal covering, then $G(\tilde{X}) \cong \pi_1(X, x_0)/H$. Thus if $\tilde{X}$ is the universal covering, then $G(\tilde{X}) \cong \pi_1(X, x_0)$.*

Thus if we have a group Γ acting properly discontinuously on a simply-connected, locally path-connected space M, a base point $x_0 \in M$ and we have the quotient map $p : M \to M/\Gamma$, then by Corollary 5.19 we have that $\pi_1(M/\Gamma, p(x_0)) \cong \Gamma$.

5.3 Cayley graph and Cayley complex

In the last section we saw that we can realize any group G as a fundamental group of some space. More precisely, given any group G we are going to construct a simply-connected topological space X, such that G acts free and proper discontinuously. Then by Corollary 5.19, we have that G is the fundamental group of X/G.

To construct this space, we proceed as follows: Let G be a finitely generated and finitely presentable group, let $S = \{c_1, \ldots, c_k\}$ be the generating set of G.

Let's consider $A = S \cup S^{-1}$, where $S^{-1} = \{c_1^{-1}, \ldots, c_k^{-1}\}$ is the set of formal inverses of the generating set S (if there is an element $a \in S$ such that $a^2 = 1$, we take $a^{-1} \in S^{-1}$ as formal inverse). Let

$$G = \langle A | R_1 = \cdots = R_p = 1 \rangle$$

be a presentation of G, where R_i are relations on elements of A and consider the involution $\iota : A \to A$ given by $\iota(c_j) = \overline{c_j}$ for $j = 1, \ldots, k$ where $\overline{c_j} = c_j^{-1}$ as element of G. We call this presentation admissible.

Definition 5.20. Suppose that we have an admissible presentation of the group G. Then, the Cayley graph of G respect to the presentation is given by $C(G) = (V, E, \iota)$, where:

- The set of vertices is given by $V = G$.
- Two vertex $g, h \in G$ are connected by an edge if $g^{-1}h \in A$. Since G is a group, then g and h are connected if and only if $h = ga$ for $a \in A$. Thus, we say that h and ga are connected by a directed edge labelled by a.
- The involution $\iota : A \to A$ is the involution which takes the edge, which connects h and g labelled by a, with the edge which connects g and h labelled by a^{-1}.

Example 5.21. Let F be a free group over the set X. Then, F has a presentation

$$F = \langle \{x\}_{x \in X} | \emptyset \rangle.$$

In order to have an admissible presentation, we add a generator x^{-1} for each $x \in X$ and we have

$$F = \langle \{x, x^{-1}\}_{x \in X} | xx^{-1} = x^{-1}x = e \rangle.$$

Thus the vertices on $C(F)$ are labelled by the reduced words over the set of generators $\{x, x^{-1}\}_{x \in X}$, where a reduced word is a word in this letters without subwords of the form xx^{-1} for $x \in X$. There is a geometric edge labelled by xx^{-1} between $x_1x_2 \cdots x_kx$ and $x_1x_2 \cdots x_k$, where $x_1, \ldots, x_k \in X$. The corresponding Cayley graph is a tree and, hence, its geometric realisation $|C(F)|_\iota$ is simply-connected (see Figure 25):

Let $R_j, 1 \leq j \leq p$ be a relation in G, written as $R_j = a_{j_1} \cdots a_{j_k}$, where $a_{j_i} \in A$. Then any $g \in G$ satisfies

$$gR_j = g(a_{j_1} \cdots a_{j_k}) = g,$$

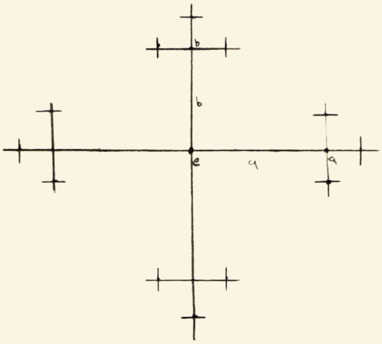

Figure 25: Cayley graph of F_2.

thus there is a loop in $C(G)$ starting and ending at g consisting of edges labelled by $a_{j_1}, \dots, a_{j_k}$ precisely in that order. In the geometric realization of $C(G)$ this loops are homeomorphic to circles and we can glue discs along this circles. The resulting space is called the Cayley 2-complex of G whit respect the given presentation and denoted by $C_2(G)$.

Example 5.22. Let's consider the group $\mathbb{Z}\times\mathbb{Z}$ with the admissible presentation:

$$\langle a, b, a^{-1}, b^{-1} | aba^{-1}b^{-1} = aa^{-1} = bb^{-1} = e\rangle,$$

then $C_2(\mathbb{Z}\times\mathbb{Z})$ is as shown in the following figure.

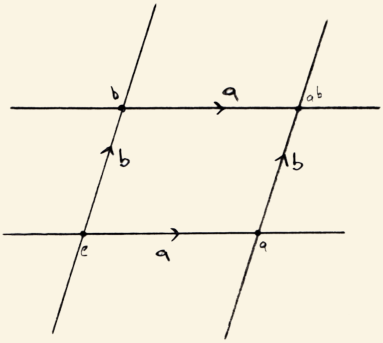

Figure 26: Cayley 2-complex of $\mathbb{Z}\times\mathbb{Z}$.

Observe that G acts on itself by left action, thus G acts on the set of vertices of $C(G)$. Extend this action into an action on $C(G)$ in the following way: if $k \in G$, then we send the edge which connects g and h to the edge which connects kg and kh. This is well-defined because if $g^{-1}h \in A$, then $(kg)^{-1}kh = g^{-1}k^{-1}kh = g^{-1}h \in A$. This action induces an action on the geometric realization $|C(G)|_\iota$

and extends to $C_2(G)$ by sending a disc attached to the loop corresponding to a relation R_j, to the disc attached to the loop corresponding to a relation kR_j. The action of G on $C(G)$ (see [3]) is free, transitive and if we have a neighborhood small enough, we will have at most the number of elements in a cycle satisfying $U \cap g(U) \neq \emptyset$. Since the cycles are finite, then we have that the action is proper discontinuous. Thus, we have the following proposition:

Proposition 5.23. *If G is a group generated by S, then $C_2(G)$ is the universal covering of X_G, where X_G is the space with $\pi_1(X_G) \cong G$ constructed by taking a wedge of circles, one for each generator in $S \cup S^{-1}$, and attaching a disc for each relation.*

Proof. Let $p : C_2(G) \to C_2(G)/G$ by the quotient map given by identify the orbits of the action of G on $C_2(G)$. Since $C_2(G)$ is arc-connected and locally arc-connected, since $S \cup S^{-1}$ is a generating set of G, then by Corollary 5.19:

$$G \cong \pi_1(C_2(G)/G)/p_*(\pi_1(C_2(G))).$$

Therefore, if we prove that $p_*(\pi_1(C_2(G)))$ is trivial, then we have

$$G \cong \pi_1(C_2(G)/G).$$

To do this, we first identify $C_2(G)/G$ as X_G. Note that every vertex is identified in $C_2(G)/G$, because every group element is sent to any other group element, because $S \cup S^{-1}$ is a generating set of G. Since every vertex in $C(G)$ has $|S|$ edges attached to it (one for every element of S), then we see that $C_2(G)/G$ is a wedge of $|S|$ many circles with discs attached to corresponding relations on the generators. This is exactly X_G. Thus,

$$C_2(G)/G \cong X_G.$$

Therefore, from above we also have:

$$G \cong \pi_1(X_G)/p_*(\pi_1(C_2(G))).$$

However, X_G is constructed, such that $\pi_1(X_G) \cong G$. It follows from

$$\pi_1(X_G) \cong G \cong \pi_1(X_G)/p_*(\pi_1(C_2(G)))$$

that $p_*(\pi_1(C_2(G)))$ is trivial. Since $p : C_2(G) \to C_2(G)/G = X_G$ is a covering, then $p_* : \pi_1(C_2(G)) \to \pi_1(X_G)$ is injective. Hence, $\pi_1(C_2(G))$ is trivial by above, and $C_2(G)$ is the universal covering for X_G. □

5.4 Classification of surfaces II: Unicity

We apply the last proposition to prove the part of unicity of the Classification Theorem.

Theorem 5.24. *The fundamental group of a surface S_g is given by*

$$\pi_1(S_g) \cong \langle a_1, b_1, \ldots, a_g, b_g | \Pi_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} = e \rangle.$$

These groups are non-isomorphic for different choices of g.

Proof. For $g = 0$, we have that $\pi_1(S_0) = \{e\}$ since S_0 is simple-connected. Let's assume that $g \geq 1$ and we define

$$A_g := \langle a_1, b_1, \ldots, a_g, b_g | \Pi_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} = e \rangle.$$

If we attach the inverse of the generators to this presentation and we construct the associated Cayley graph $C(A_g)$ and the Cayley 2-complex $C_2(A_g)$. Then by the Proposition 5.21, we have that $C_2(A_g)/A_g$ is homeomorphic to a wedge of circles labelled by $a_1, b_1, a_1^{-1}, b_1^{-1}, \ldots, a_g, b_g, a_g^{-1}, b_g^{-1}$, with a disc attached to them. This description is the same as the surface S_g; therefore, by Proposition 5.21 we have that $\pi_1(S_g) = A_g$. □

6 Combinatorial description of the fundamental group using ribbon graphs

In this section, we relate the filling ribbon graph of a surface and its fundamental group. We know that given a surface S exists a filling ribbon graph. This ribbon graphs are not unique, but we can deform this graphs to a ribbon graph of type Γ_g. Using different ribbon graphs, we can compute the fundamental group. This gives us different presentations of the fundamental group.

Let Γ be a filling ribbon graph. Denote by E, V and F the set of edges, vertices, and faces respectively.

Definition 6.1 (Combinatorial paths and loops)**.** A discrete path is a finite sequence $(e_1, \ldots, e_n)$ of edges, such that $e_i^+ = e_{i+1}^-$. The starting point of such path is e_1^+ and the ending point is e_n^+. A path is a discrete loop if its starting point and ending point are the same. We say that a loop has a base point at v_0, if v_0 is the starting and ending point. The inverse path of $e = (e_1, \ldots, e_n)$ is $\bar{e} = (\overline{e_n}, \ldots, \overline{e_1})$.

Let $F(E)$ be the free group generated by the set of edges; note that every path defines an element of $F(E)$. Let $L_\Gamma^{v_0}$ be the image of the loops with base point v_0 on $F(E)$, then $L_\Gamma^{v_0}$ is a subgroup of $F(E)$. Let $R_\Gamma^{v_0}$ be the subgroup of $L_\Gamma^{v_0}$ normally generated by the faces $f = (e_1, \ldots, e_k)$.

Definition 6.2. Let Γ be a filling ribbon graph, then:

- The group $L_\Gamma^{v_0}$ is the group of loops with base point v_0.
- The group $R_\Gamma^{v_0}$ is the group of homotopically trivial loops.
- The ribbon fundamental group is $\hat{\pi}_1(\Gamma, v_0) = L_\Gamma^{v_0}/R_\Gamma^{v_0}$.
- Two paths e and f with the same starting and final point are homotopics if the loop $e\bar{f}$ is an element of $R_\Gamma^{v_0}$.

Now we relate the fundamental group of a surface S with the ribbon fundamental group of its ribbon graph.

Theorem 6.3. *Let S be a closed surface and $i : |\Gamma|_I \to S$ be an embedding of the geometric realization of a filling ribbon graph Γ on S. Then, the natural mapping $i_* : \hat{\pi}_1(\Gamma, v_0) \to \pi_1(S, v_0)$, which send every combinatorial loop to its geometric realization, is an isomorphism of the ribbon fundamental group and the fundamental group of the surface.*

Proof. Since Γ can be deformed to a filling ribbon graph Γ_g which is a wedge of circles, we have that $\hat{\Pi}_1(\Gamma, v_0) = \hat{\pi}_1(\Gamma_g, v_0) = A_g$, this by Proposition 5.21. By Theorem 5.22, we have that $\pi_1(S_g) = \pi_1(S) = A_g$. Therefore, we have the following:

$$\hat{\pi}_1(\gamma, v_0) = A_g = \pi_1(S)$$

and we are done. □

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Complex Schottky groups cannot act on $\mathbb{P}^{2n}_{\mathbb{C}}$ as a subgroup of $PSL(2n+1, \mathbb{C})$, an alternative proof *

Vanessa Alderete †

Abstract

Classical Schottky groups are discrete subgroups of $PSL(2, \mathbb{C})$ acting on the Riemann sphere by inversions on circles. The complex Schottky groups are a generalization of Schottky groups, these are discrete subgroups of $PSL(n+1, \mathbb{C})$ acting on $\mathbb{P}^n_{\mathbb{C}}$ with a dinamical behavior reminiscent to the dynamics of the classical Schottky groups. They were defined by M.V. Nori [12] in 1986, and studied in more detail in 2002 [14] by J. Seade and A. Verjovsky.

In 2008, A. Cano [4] proved that in Complex Kleinian groups there is no group Γ of $PSL(2n+1, \mathbb{C})$ acting on $\mathbb{P}^{2n}_{\mathbb{C}}$ as a complex Schottky group.

Later in 2016 [1], the authors proved that a condition for a group Γ of $PU(k, l)$ acts as a complex Schottky group on $\mathbb{P}^n_{\mathbb{C}}$, with $n+1 = k+l$, that is, $l = k$. It means that we have the dynamics of complex Schottky groups for elements in $PU(k, l)$ only in even dimension.

The objective of this work is to give an alternative proof of Cano's theorem, but this time using the tools used in [1] that are more geometric.

Keywords: Complex Schottky groups, complex hyperbolic spaces.

1 Introduction

Kleinian groups are discrete subgroups of $PSL(2, \mathbb{C})$ acting by Möbius transformations on the Riemann sphere. In those there exist a special type of groups called *Schottky groups*; these are costructed by invertions on Jordan's curves on the Riemann sphere.

An important aplication of Schottky groups is the Koebe's Retrosection Theorem where he proved that any compact manifold $\tilde{M}$ of genus n can be represented by the quotient Ω/Γ, where Ω is a Γ-invatiant open set in the Riemann sphere, and Γ is a Schottky group.

In 1986 [12], M.V. Nori gave a generalization of Schottky groups to higher dimensions. Later in [13, 15, 16], J. Seade and A. Verjovsky gave a generalization

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of Kleinian groups to higher dimensions acting on the projective space and called them *Complex Kleinian groups*. Then in [14] J. Seade and A. Verjovsky gave another constructive generalization to get a Schottky group Γ, as a discrete subgroup of $PSL(n+1,\mathbb{C})$ acting on $\mathbb{P}^n_{\mathbb{C}}$, and they call these kind of groups *complex Schottky groups*.

Nowadays the complex Schottky groups still poses interesting open questions, as we can see in [1] and [10].

In 2008, Á. Cano [4] proved that actually the complex Schottky groups cannot act on $\mathbb{P}^{2n}_{\mathbb{C}}$ as subgroups of $PSL(2n+1,\mathbb{C})$, given a proof using algebraic and dinamical tools. In [1], the authors proved that if we take Γ to be a subgroup of $PU(k,l)$ acting on an Hermitian space as a complex Schottky group, then k have to be iqual to l.

The objective of this work is to give a proof of the result given by Á. Cano in [4], but using the geometric tools used in [1]. Formally the theorem is the following:

Theorem 1.1. *Let Γ be a discrete subgroup of $PSL(2n+1,\mathbb{C})$, then Γ cannot act as a complex Schottky group on $\mathbb{P}^{2n}_{\mathbb{C}}$.*

This paper is organized as follows: in Section 2, we review some general facts and introduce the notation used along the text, and in Section 3, we give some useful tools that we will use to prove the main results of this article also given in this section.

2 Preliminaries

2.1 Projective Geometry

We will work with the complex projective space $\mathbb{P}^n_{\mathbb{C}}$. To define subspaces of $\mathbb{P}^n_{\mathbb{C}}$, we consider the quotient map $[\ ]:\mathbb{C}^{n+1}\setminus\{0\}\to\mathbb{P}^n_{\mathbb{C}}$, then a non-empty set $H\subset\mathbb{P}^n_{\mathbb{C}}$ is said to be a projective subspace of dimension k if there is a $\mathbb{C}$-linear subspace $\widetilde{H}$ of dimension $k+1$, such that $[\widetilde{H}\setminus\{0\}]=H$. In this article, $\{e_1,\ldots,e_{2n+1}\}$ will denote the standard basis for $\mathbb{C}^{2n+1}$.

Given a set of points S in $\mathbb{P}^n_{\mathbb{C}}$, we define:

$$Span(S)=\bigcap\{P\subset\mathbb{P}^n_{\mathbb{C}}\mid P \text{ is a projective subspace containing } S\}.$$

Clearly, $Span(S)$ is a projective subspace of $\mathbb{P}^n_{\mathbb{C}}$.

2.2 Projective and Pseudo-projective Transformations

Every linear isomorphism of $\mathbb{C}^{n+1}$ defines a holomorphic automorphism of $\mathbb{P}^n_{\mathbb{C}}$. Also, it is well-known that every holomorphic automorphism of $\mathbb{P}^n_{\mathbb{C}}$ arises in this way. The group of *projective automorphisms* of $\mathbb{P}^n_{\mathbb{C}}$ is defined by

$$PSL(n+1,\mathbb{C}) := SL(n+1,\mathbb{C})/\mathbb{C}^*,$$

where $\mathbb{C}^*$ acts by the usual scalar multiplication. Then, $PSL(n+1,\mathbb{C})$ is a Lie group whose elements are called *projective transformations*.

We denote by $[[\;]] : SL(n+1,\mathbb{C}) \to PSL(n+1,\mathbb{C})$ the quotient map. Given $\gamma \in PSL(n+1,\mathbb{C})$, we say that $\widetilde{\gamma} \in SL(n+1,\mathbb{C})$ is a lift of γ if $[[\widetilde{\gamma}]] = \gamma$.

A way to work with projective transformation is precisely using liftings. One important tool to work with is the Polar Decomposition, or its equivalent, the Singular Value Decomposition for elements on $SL(2n+1,\mathbb{C})$; for details see [17].

Considering the notation used in [17], we denote by $HPD(n)$ the group of positive defined Hermitian matrices and by $U(n)$ the group of unitary matrices, both in $GL(n,\mathbb{C})$.

Theorem 2.1 (Polar Decomposition). *Given a matrix $M \in GL(n,\mathbb{C})$, there exist a unique pair:*

$$(H,Q) \in HPD(n) \times U(n),$$

such that $M = HQ$.

The map $M \mapsto (H,Q)$ is called the Polar Decomposition of M and it is an homeomorphism between $GL(n,\mathbb{C})$ and $HPD(n) \times U(n)$.

From the fact that for all positive defined matrices H there exist a positive defined matrix h, such that $h^2 = H$, we have that starting with the Polar Decomposition of a matrix M, we can obtain the Singular Value Decomposition given in the next theorem.

Theorem 2.2 (Singular Value Decomposition). *Given a matrix $M \in GL(n,\mathbb{C})$, there exist two unitary matrices $U, V \in U(n)$ and a diagonal matrix:*

$$\mathcal{D}(M) = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix},$$

such that $M = U\mathcal{D}(M)V$ and where $\lambda_1, \ldots, \lambda_n \in (0,+\infty)$. The $\lambda_i's$ are called the singular values of M, they are the square roots of the eigenvalues of the matrix H given in Theorem 2.1 and they are uniquely defined up to permutation.

Actually we can order the $\lambda_i's$, such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n > 0$.

The last decomposition works even for non-square matrices.

2.3 Pseudo-projective Transformations

The space of linear transformations from $\mathbb{C}^{n+1}$ to $\mathbb{C}^{n+1}$, denoted by $M(n+1,\mathbb{C})$, is a linear complex space of dimension $(n+1)^2$. Note that $GL(n+1,\mathbb{C})$ is an open dense set of $M(n+1,\mathbb{C})$. Hence, $PSL(n+1,\mathbb{C})$ is an open dense set in $QP(n+1,\mathbb{C}) = (M(n+1,\mathbb{C}) \setminus \{0\})/\mathbb{C}^*$; the latter is called *the space of pseudo-projective maps*. Let $\widetilde{M} : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ be a non-zero linear transformation and $Ker(\widetilde{M})$ be its kernel. We denote by $Ker([[\widetilde{M}]])$ the respective projectivization. Then, $\widetilde{M}$ induces a well defined map $[[\widetilde{M}]] : \mathbb{P}^n_{\mathbb{C}} \setminus Ker([[\widetilde{M}]]) \to \mathbb{P}^n_{\mathbb{C}}$ given by

$$[[\widetilde{M}]]([v]) = [\widetilde{M}(v)] \,.$$

The following proposition shows that we can find sequences in $QP(n+1,\mathbb{C})$, such that the convergence as a sequence of points in a projective space coincides with the convergence as a sequence of functions on $QP(n+1,\mathbb{C})$.

Proposition 2.3 (See [3]). *Let $(\gamma_m) \subset PSL(n+1,\mathbb{C})$ be a sequence of distinct elements, then:*

1. *There is a subsequence $(\tau_m) \subset (\gamma_m)$ and $\tau_0 \in M(n+1,\mathbb{C}) \setminus \{0\}$, such that $\tau_m \xrightarrow[m\to\infty]{} \tau_0$ as points in $QP(n+1,\mathbb{C})$.*

2. *If (τ_m) is the sequence given by the previous part of this lemma, then $\tau_m \xrightarrow[m\to\infty]{} \tau_0$ as functions, uniformly on compact sets of $\mathbb{P}^n_{\mathbb{C}} \setminus Ker(\tau_0)$.*

The following lemma will be used at the proof of the main result; for further details see [2].

Lemma 2.4. *Let $(\gamma_m), (\tau_m) \subset PSL(n+1,\mathbb{C})$ be sequences, such that $\gamma_m \xrightarrow[m\to\infty]{} \gamma_0$ and $\tau_m \xrightarrow[m\to\infty]{} \tau_0$. If $Im(\tau) \cap Ker(\gamma) \neq \emptyset$, then*

$$\gamma_m \tau_m \xrightarrow[m\to\infty]{} \gamma_0 \tau_0 .$$

Now we recall the classification of projective transformations on $PSL(n+1,\mathbb{C})$. (See [6]):

Definition 2.5. Let $\gamma \in PSL(n+1,\mathbb{C})$, then γ is said to be:

1. *Loxodromic* if γ has a lift $\widetilde{\gamma} \in SL(n+1,\mathbb{C})$, such that $\widetilde{\gamma}$ has at least one eigenvalue outside the unit circle.

2. *Elliptic* if γ has a lift $\widetilde{\gamma} \in SL(n+1,\mathbb{C})$, such that $\widetilde{\gamma}$ is diagonalizable and all of its eigenvalues are in the unit circle.

3. *Parabolic* if γ has a lift $\widetilde{\gamma} \in SL(n+1,\mathbb{C})$, such that $\widetilde{\gamma}$ is non-diagonalizable and all of its eigenvalues are in the unit circle.

2.4 The Grassmanians and the Plücker embedding

Let $0 \leq k < n$, we define the Grassmanian $Gr(k,n)$ as the space of all k-dimensional projective subspaces of $\mathbb{P}^n_{\mathbb{C}}$ endowed with the Hausdorff topology. One has that $Gr(k,n)$ is a compact, connected complex manifold of dimension $k(n-k)$. A method to realize the Grassmanian $Gr(k,n)$ as a subvariety of the projective space of the $(k+1)$-th exterior power of $\mathbb{C}^{n+1}$, in symbols $\mathbb{P}(\bigwedge^{k+1}\mathbb{C}^{n+1})$, is done by the so called Plücker embedding, which is given by

$$\begin{aligned} \iota : Gr(k,n) &\to \mathbb{P}(\textstyle\bigwedge^{k+1}\mathbb{C}^{n+1}) \\ \iota(V) &\mapsto [v_1 \wedge \cdots \wedge v_{k+1}], \end{aligned}$$

where $Span(\{v_1, \cdots, v_{k+1}\}) = V$. We can induce an action of $PSL(n+1,\mathbb{C})$ on $Gr(k,n)$ and $\mathbb{P}(\bigwedge^{k+1}\mathbb{C}^{n+1})$ as follows:

Let $[[T]] \in PSL(n+1,\mathbb{C})$, take $W = Span(\{w_1, \ldots, w_{k+1}\}) \in Gr(k+1, n+1)$ and a point $w = [w_1 \wedge \cdots \wedge w_{k+1}] \in \mathbb{P}(\bigwedge^{k+1}\mathbb{C}^{n+1})$. Now set

$$T(W) = Span([[T]](w_1), \ldots, [[T]](w_{k+1}))$$

and

$$\bigwedge^{k+1} T(w) = [T(w_1) \wedge \cdots \wedge T(w_{k+1})],$$

then we have the following commutative diagram:

$$\begin{array}{ccc} Gr(k,n) & \xrightarrow{\;T\;} & Gr(k,n) \\ \downarrow \iota & & \downarrow \iota \\ \mathbb{P}(\bigwedge^{k+1}\mathbb{C}^{n+1}) & \xrightarrow{\bigwedge^{k+1} T} & \mathbb{P}(\bigwedge^{k+1}\mathbb{C}^{n+1}). \end{array} \tag{1}$$

2.5 Complex Schottky groups

Complex Schottky groups are defined as follows; compare with definitions in [7, 8, 9, 12, 13].

Definition 2.6 (See [4]). Let $\Gamma \subset PSL(n+1,\mathbb{C})$, we say that Γ is a *complex Schottky group* acting on $\mathbb{P}^n_{\mathbb{C}}$ with g generators if:

1. There are $2g$, for $g \geq 2$, open sets $R_1, \ldots, R_g$, $S_1, \ldots, S_g$ satisfying the following property:

   (a) Each of these open sets is the interior of its closure.

   (b) The closures of the $2g$ open sets are pairwise disjoint.

2. The group has a generating set $\{\gamma_1, \ldots, \gamma_g\}$ with the property $\gamma_j(R_j) = \mathbb{P}^n_{\mathbb{C}} \setminus \overline{S_j}$ for each j.

An important theorem that caracterizes the complex Schottky groups was proved by Á. Cano and is the following:

Theorem 2.7 (See [4]). *Let $\Gamma \subset PSL(n+1,\mathbb{C})$ be a complex Schottky group with g generators, then Γ is a purely loxodromic free group with g generators. If $D = \bigcap_{j=1}^{g} \mathbb{P}^n_{\mathbb{C}} \setminus (R_j \cup S_j)$, then $\Omega_\Gamma = \Gamma D$ is a Γ-invariant open set where Γ acts properly discontinuously. Moreover, Ω_Γ has compact quotient and the limit set $\Lambda_S(\Gamma) = \mathbb{P}^n_{\mathbb{C}} \setminus \Omega_\Gamma$ is disconnected.*

The set $\Lambda_S(\Gamma)$ is called the *Schottky limit set* of Γ.

3 The alternative prove

3.1 Some auxiliar tools

In this section, we give a series of lemmas and definitions that allow us to understand the convergence of sequences of distinct elements of $PSL(2n+1,\mathbb{C})$ acting on the projective space $\mathbb{P}^{2n}_{\mathbb{C}}$.

Lemma 3.1 ([6]). *Let $\gamma \in PSL(n,\mathbb{C})$ be a non-elliptic element. If there is a sequence $(n_m) \subset \mathbb{Z}$ of distinct elements, such that there is a point p and a hyperplane $\mathcal{H}$ satisfying $\gamma^{n_m} \xrightarrow[m\to\infty]{} p$ uniformly on compact sets of $\mathbb{P}^{n-1}_{\mathbb{C}} \setminus \mathcal{H}$, then p is a fixed point of γ.*

Lemma 3.2 ([1]). *Let $([[T_m]])$ be a sequence of different elements of $PSL(k+l,\mathbb{C})$, such that there is a point $p = [w_1 \wedge \cdots \wedge w_k]$ and a hyperplane $\mathcal{H}$ satisfying $[[\wedge^k T_m]] \xrightarrow[m\to\infty]{} p$ uniformly on compact sets of $\mathbb{P}(\wedge^k(\mathbb{C}^{k+l})) \setminus \mathcal{H}$. Then for all $U \in Gr(k,k+l) \setminus \iota^{-1}(\mathcal{H})$, we have that $T_m(U)$ converges to $W = Span(w_1,\ldots,w_k)$ in $\mathbb{P}^{k+l}_{\mathbb{C}}$ in the Hausdorff topology.*

Definition 3.3. Let (γ_m) be a sequence in a topological space X, we say that (γ_m) is a *divergent sequence* if (γ_m) leaves every compact set of X.

The following definition tell us when a sequences converge simply to infinity and it will be useful to prove the lemma **??**.

Definition 3.4. Let $(\gamma_m) \subset PSL(2n+1,\mathbb{C})$ be a divergent sequence and consider the Singular Value Decomposition of each γ_m. We say that (γ_m) *converges simply to infinity* if the following are satisfy:

1. The compact factors in the Singular Value Decomposition U_m and V_m converge to some U and V in $U(2n+1)$.

2. There exist t natural numbers $k_1, ..., k_t \in \mathbb{N}$, such that $k_1 + \cdots + k_t = 2n+1$, t sequences $(\lambda_{1_m}), ..., (\lambda_{t_m}) \subset \mathbb{R}$, and t block matrices $D_{1_m} \in SL(k_1, \mathbb{R}), ..., D_{t_m} \in SL(k_t, \mathbb{R})$, satisfying:

$$\mathcal{D}_m(\gamma_m) = \begin{pmatrix} \lambda_{1_m} D_{1_m} & & \\ & \ddots & \\ & & \lambda_{t_m} D_{t_m} \end{pmatrix},$$

for each m, where the rates $\lambda_{i_m}/\lambda_{j_m} \to \infty$ when $m \to \infty$, for all $i > j$, and the block matrices D_{i_m} converge to some $D_i \in SL(k_i, \mathbb{R})$ as $m \to \infty$.

Definition 3.5. Let $x \in \mathbb{P}^{2n}_{\mathbb{C}}$ and (γ_m), a divergent sequence of different elements in $PSL(2n+1, \mathbb{C})$, we define $\mathfrak{D}_{(\gamma_m)}(x)$ as the set of all the accumulation points of sequences of the form $(\gamma_m(x_m))$, where (x_m) is a sequence that converges to x in $\mathbb{P}^{2n}_{\mathbb{C}}$.

An useful tool has been *the λ-lemma*, it has been used in distinct contexts for example: Frances in [7] for the group $O(n)$, J.P. Navarrete [11] for $PU(2,1)$, Á. Cano- B. Liu- M. López for the group $PU(1,n)$ [5], and M. Méndez [9] for the group $PU(k,l)$. In this paper, we give a version for the group $SL(2n+1, \mathbb{C})$.

First, let us give an intuitive idea of how the λ-lemma works. Consider an action of a divergent sequence (γ_m) of different elements in $PSL(2n+1, \mathbb{C})$ on $\mathbb{P}^{2n}_{\mathbb{C}}$ and take the Singular Value Decomposition of (γ_m) for all m, then the λ-lemma gives us a partition of $\mathbb{P}^{2n}_{\mathbb{C}}$ into projective subspaces and with this allows us to understand the set $\mathfrak{D}_{(\gamma_m)}(x)$.

A virtue of the λ-lemma is that even if we change the context in which we are working, the proof of it is essentialy the same for all the groups.

Lemma 3.6 (λ-lemma). *Let $(\gamma_m) \subset SL(2n+1, \mathbb{C})$ be a sequence tending simply to infinity, then there exist:*

- *t natural numbers $k_1, ..., k_t \in \mathbb{N}$,*
- *$(2t)$ pairs of projective subespaces $P^+_1, ..., P^+_t, P^-_1, ..., P^-_t$,*
- *a set of projective transformations $\gamma_i : P^-_i \to P^+_i$, and*
- *a pseudo-projective transformation $\gamma \in QP(2n+1, \mathbb{C})$,*

such that:

1. $Im(\gamma) = P_1^+$, *and* $Ker(\gamma) = Span\left(\bigcup_{i=2}^{t} P_i^-\right)$.

2. $dim\left(Span\left(\bigcup_{i=1}^{t} P_i^{\pm}\right)\right) = t + \sum_{i=1}^{t} dim(P_i^{\pm}) = 2n+1$.

3. *One of the following holds:*

   (a) *If* $x \in \mathbb{P}_{\mathbb{C}}^{2n} \setminus Ker(\gamma)$, *then* $(\gamma_m) \to \gamma$ *as* $m \to \infty$, *and*

   $$\mathcal{D}_{(\gamma_m)}(x) = \gamma(x).$$

   (b) *If* $j \in \{2, t-1\}$, $y \in P_j^-$ *and*

   $$x \in Span\left(\{y\}, \left(\bigcup_{i=j+1}^{t} P_i^-\right)\right) \setminus \left(\bigcup_{i=j+1}^{t} P_i^-\right),$$

   then

   $$\mathcal{D}_{(\gamma_m)}(x) = Span\left(\{\gamma_j(y)\}, \left(\bigcup_{i=1}^{j-1} P_i^+\right)\right).$$

   (c) *If* $x \in P_t^-$, *then*

   $$\mathcal{D}_{(\gamma_m)}(x) = Span\left(\{\gamma_t(y)\}, \left(\bigcup_{i=1}^{t-1} P_i^+\right)\right).$$

Observe that we can also consider the λ-lemma for γ^{-m}, using the fact that $D(\gamma_m)$ is diagonal and invertible.

Here we only give the key parts of the proof and refer to [5], [9] or [1] for details.

Proof. Let (γ_m) be a divergent sequence of different elements of $PSL(2n+1, \mathbb{C})$. By the Singular Value Decomposition, we have that for each m exist $U_m, V_m \in U(2n+1)$, such that:

$$\gamma_m = U_m \begin{pmatrix} \lambda_{1_m} D_{1_m} & & \\ & \ddots & \\ & & \lambda_{t_m} D_{t_m} \end{pmatrix} V_m.$$

Now, following with the notation given in Theorem 2.2, define the projective subspaces:

$$P_j = Span\{e_{\sum_1^{j-1} k_j+1}, ..., e_{\sum_1^{j} k_j}\},$$

with $1 \leq j \leq t$. Then, we define $P_j^+ = [UP_j]$, and $P_j^- = [V^{-1}P_j]$.

Since here we can define the projective transformations γ_j as a traslations given by V and U, which are the limits of the sequences V_m and U_m. Also we can define the pseudo-projective transformation γ as the projective transformation, whose image is just the projective subspace P_1^+.

Now, the first part of the proof follows by the previous. The second part are straightforward computations about dimension. For the third part, we just have to observe that the behavior of the dynamics of γ_m is determined by the values on the diagonal of $\mathcal{D}(\gamma_m)$ and see to where it converges. □

Now, we are able to prove the main theorem:

Theorem 3.7. *Let Γ be a discrete subgroup of $PSL(2n+1,\mathbb{C})$, then Γ cannot acts as a complex Schottky group on $\mathbb{P}_{\mathbb{C}}^{2n}$.*

Proof. Let us proceed by contradiction. Suppose that $\Gamma \subset PSL(2n+1,\mathbb{C})$ acts as a complex Schottky group on $\mathbb{P}_{\mathbb{C}}^{2n}$. Take a generator $\gamma \in \Gamma$ and let $\tilde{\gamma} \in SL(2n+1,\mathbb{C})$ be a lift of γ. Consider the Singular Value Decomposition of $\tilde{\gamma}^m$, then we obtain sequences (U_m) and (V_m) in $U(2n+1)$, and $(\mathcal{D}_m(\gamma_m))$ in $SL(2n+1,\mathbb{C})$ satisfying $\tilde{\gamma}^m = U_m(\mathcal{D}_m(\gamma_m))V_m$.

Since (U_m) and (V_m) lie in a compact set, there is a subsequence $(m_s) \subset (m)$ and elements $\bar{U}$ and $\bar{V}$ in $U(2n+1)$, such that U_{m_s} converges to $\bar{U}$ and V_{m_s} converges to $\bar{V}$.

Now, for each m consider the block decomposition of $(\mathcal{D}_m(\gamma_m))$ as in Definition 3.4.2,

$$\mathcal{D}_m(\gamma_m) = \begin{pmatrix} \lambda_{1_m} D_{1_m} & & \\ & \ddots & \\ & & \lambda_{t_m} D_{t_m} \end{pmatrix};$$

in this way, we have that $\lambda_{1_m} > ... > \lambda_{t_m} > 0$.

Clearly, we can assume that (γ_m) tends simply to infinity.

We claim that there exist projective subspaces P and Q, satisfying the following properties:

1. The spaces P, Q are invariant under the action of γ. Moreover, P is attracting and Q is repelling.

2. If R_γ, S_γ are the disjoint open sets associated to γ given in the definition of a complex Schottky group, then either $P \subset R_\gamma$ and $Q \subset S_\gamma$, or $Q \subset R_\gamma$ and $P \subset S_\gamma$. In particular, it follows that P and Q are also disjoint and lie in distinct connected components of $\Lambda_S(\Gamma)$.

3. The dimensions satisfy $dimP < n$ and $dimQ < n$.

4. If we defined $\hat{P}$ as the complementary space of Q and $\hat{Q}$ as the complementary space of P, we have that $\hat{P} \not\subset \Lambda_S(\Gamma)$ and $\hat{Q} \not\subset \Lambda_S(\Gamma)$.

Set P and Q the projectivizations of the spaces $P' = \hat{U}(Span(\{e_1, ..., e_{k_1}\}))$ and $Q' = \bar{V}^{-1}(Span(\{e_{((2n+1)-k_t+1)}, ..., e_{2n+1}\}))$.

Let us show the first part (1), consider the action of $\wedge^{k_1} D(\gamma_m)$ on $\wedge^{k_1}\mathbb{C}^{2n+1}$, then a straightforward calculation shows that the matrix of $\wedge^{k_1} D(\gamma_m)$ with respect the standard ordered basis β of $\wedge^{k_1}\mathbb{C}^{2n+1}$ is given by

$$A_m = \begin{pmatrix} \theta_1 & & & \\ & \theta_2 & & \\ & & \ddots & \\ & & & \theta_{\binom{k}{n}} \end{pmatrix},$$

where θ_i is the product of k_i elements taken from the set $\{e^{\lambda_{i,m}(\gamma^{nm})}\}$ and ordered in the lexicographical order in (i, m). In fact, $\theta_1 > \theta_2 > ... > \theta_{\binom{k_1}{2n+1}}$. Hence, $[[\mathcal{D}(\gamma_m)]]$ converges to $x = [e_1 \wedge \cdots \wedge e_{k_1}]$ uniformly on compact sets of $\mathbb{P}(\wedge^{k_1}(\mathbb{C}^{2n+1})) \setminus Span(\beta \setminus \{x\})$.

Therefore, by Lemma 2.4, we conclude that $[[\wedge^{k_1}\tilde{\gamma}^m]]$ converges to the point $[[\wedge^k U]][e_1 \wedge \cdots \wedge e_{k_1}]$ uniformly on compact sets of $\mathbb{P}(\wedge^{k_1}(\mathbb{C}^{2n+1})) \setminus [\wedge^{k_1} V^{-1}]Span(\beta \setminus \{x\})$. Finally, from Lemma 3.1, we conclude that x is a fixed point of $[[\wedge^{k_1}\tilde{\gamma}^{m_s}]]$, in consequence, $P = [C]Span(\{[e_1], \ldots, [e_{k_1}]\})$ is attracting and invariant under γ. In a similar way, we can prove that Q is repelling and invariant.

Part (2). On the contrary, suppose there exist $x \in P \cap (\mathbb{P}^{2n}_{\mathbb{C}} \setminus (R_\gamma \cup S_\gamma)) \neq \emptyset$. Because of (1), we have that x is an atracting point, then for some $z \in \mathbb{P}^{2n}_{\mathbb{C}} \setminus (R_\gamma \cup S_\gamma)$, we have that $\gamma^m(z)$ converges to x as m tends to ∞, but in the other hand by the dynamics of Γ as a complex Schottky group, we have that $\gamma(z) \in S_\gamma$ and also $\gamma^m(z) \in S_\gamma$, then $x \in S_\gamma$, which is a contradiction.

Part (3). Suppose that $dimP = n$ and take $\gamma_1, \gamma_2 \in \Gamma$ generators of Γ and let R_i and S_i be the open set associated to γ_i with $i = 1, 2$, and suppose that $P \subset R_2$. Observe that $P' = \gamma_2^{-1}\gamma_1(P) \subset S_2$, $dimP = dimP' = n$, and $P \cap P' = \emptyset$, then $P \oplus P' = \mathbb{P}^{2n}_{\mathbb{C}}$. Now if we take the liftings of P and P', we have that $P + P'$ is a subspace of $\mathbb{C}^{2n+1}$, but $dim(P + P') = 2n + 2$, which is a contradiction. Then the dimension of P have to be less than n, this is also true for Q.

Part (4). Assume that $\hat{P} \subset \Lambda_S(\Gamma)$. By the previous part, we can assume that $P \subset S_\gamma$. Let $\gamma_1 \in \Gamma$ be a generator of Γ distinct from γ. By Lemma 3.2 we conclude that $\gamma^{-m}(\gamma_1(P))$ converges to Q; therefore, $\gamma_1^{-m}(\gamma(\hat{P}))$ converges to $\hat{Q}$. Hence, $\hat{Q} \subset \Lambda_S(\Gamma)$. Then we have that $P \subset \hat{P}$, $Q \subset \hat{Q}$, and $\hat{P} \cap \hat{Q} \neq \emptyset$, and all of these spaces are path connected, which lead us to a contradiction of (2).

Now, take the block D_j, such that the vector $e_{n+1} \in D_j$, and call L the space generated by the eigenvectors associated to the eigenvalue in D_j. Observe that $L \subset \hat{P} \cap \hat{Q}$, then by (4) $L \nsubseteq \Lambda_S(\Gamma)$; it means that $L \in \Omega_\Gamma$.

Now consider the space of lines between P and the spaces generated by the eigenvectors associated to the blocks $D_2, ..., D_{j-1}$ and call it A; also consider the space of lines between Q and the spaces generated by the eigenvectors asociated to the blocks $D_{j+1}, ..., D_t$ and call it B. Notice that A and B are connected.

To conclude the proof, by the λ-lemma, we have that if we take $p, q \in L \subset \Omega_\Gamma$ and $a \in Span(\{p\}, A) \setminus A$, and $b \in Span(\{q\}, B) \setminus B$, we have that $Span(\hat{p}, P) \cup Span(\hat{q}, Q) \subset \Lambda_\Gamma$, for some $\hat{p}, \hat{q} \in L$. But $Span(\hat{p}, P)$, $Span(\hat{q}, Q)$, and L are path connected. Then we can construct a path in Λ_Γ, passing along $\hat{p}$ and $\hat{q}$ through L and connecting P with Q, which contradict (2) and that concludes the proof of the theorem. □

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